

# A fully-discrete Semi-Lagrangian scheme for a first order mean field game problem

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## Abstract

In this work we propose a fully-discrete Semi-Lagrangian scheme for a *first order mean field game system*. We prove that the resulting discretization admits at least one solution and, in the scalar case, we prove a convergence result for the scheme. Numerical simulations and examples are also discussed.

**Keywords:** Mean field games, First order system, Semi-Lagrangian schemes, Numerical methods.

**MSC 2000:** Primary, 65M12, 91A13; Secondary, 65M25, 91A23, 49J15, 35F21.

## 1 Introduction

Initiated by the seminal work of Aumann [7], models to study equilibria in games with a large number of players have become a important research line in the fields of Economics and Applied Mathematics. In this direction, Mean Field Games (MFG) models were recently introduced by J-M. Lasry and P.-L. Lions in [20, 21, 22] in the form of a new system of Partial Differential Equations (PDEs). Under some assumptions, the solution of this system captures the main properties of Nash equilibria for differential games with a very large number of *identical “small” players*. For a survey of MFG theory and its applications, we refer the reader to [12, 18] and the lectures of P.-L. Lions at the Collège de France [24]. The evolutive PDE system introduced in [21], with variables  $(v, m)$ , is of the form:

$$\begin{aligned} -\partial_t v(x, t) - \sigma^2 \Delta v(x, t) + H(x, Dv(x, t)) &= F(x, m(t)), \quad \text{in } \mathbb{R}^d \times (0, T), \\ \partial_t m(x, t) - \sigma^2 \Delta m(x, t) - \operatorname{div}(\partial_p H(x, Dv(x, t))m(x, t)) &= 0, \quad \text{in } \mathbb{R}^d \times (0, T), \\ v(x, T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d, \quad m(0) = m_0 \in \mathcal{P}_1, \end{aligned} \tag{1.1}$$

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where  $\sigma \in \mathbb{R}$ ,  $\mathcal{P}_1$  denotes the space of probability measures on  $\mathbb{R}^d$  and  $F : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}$ ,  $G : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}$  and  $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  are given functions. The Hamiltonian  $H$  is supposed to be convex with respect to the second variable  $p$ . An important feature of the above system is its *forward-backward structure*: We have a backward Hamilton-Jacobi-Bellman (HJB) equation, i.e. with a terminal condition, coupled with a forward Fokker-Planck equation with initial datum  $m_0$ .

Under rather general assumptions, it can be proved that if  $\sigma \neq 0$  then (1.1) admits *regular solutions* (see [22, Theorem 2.6]). Based on this fact, finite differences schemes have been thoroughly analyzed in the papers [4, 1, 2]. When  $H(x, p)$  is quadratic with respect to  $p$ , specific methods have been proposed in [17, 19].

In this work, we are interested in the numerical analysis of the *first order case* ( $\sigma = 0$ ) with quadratic Hamiltonian  $H(x, p) = \frac{1}{2}|p|^2$ . In this case, system (1.1) takes the form

$$\begin{aligned} -\partial_t v(x, t) + \frac{1}{2}|Dv(x, t)|^2 &= F(x, m(t)), \quad \text{in } \mathbb{R}^d \times (0, T), \\ \partial_t m(x, t) - \operatorname{div}(Dv(x, t)m(x, t)) &= 0, \quad \text{in } \mathbb{R}^d \times (0, T), \\ v(x, T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d, \quad m(0) = m_0 \in \mathcal{P}_1. \end{aligned} \tag{1.2}$$

The second equation (i.e. the Fokker-Planck equation with  $\sigma = 0$ ) is called the *continuity equation* and describes the transport of the initial measure  $m_0$  by the flow induced by  $-Dv(\cdot, \cdot)$ . When  $F$  and  $G$  are non-local and *regularizing* operators (see [22]), the existence of a solution  $(v, m)$  of (1.2) can be proved by a fixed point argument (see [12, 24]). However, the numerical approximation of  $(v, m)$  is very challenging since, besides the forward-backward structure of (1.2), we can expect only Lipschitz regularity for  $v$  and  $L^\infty$  regularity for  $m$  (see e.g. [12]).

Although several numerical methods have been analyzed for each one of the equations in (1.2) (see e.g. the monographs [15, 29, 25] and the references therein for the HJB equation and [26, 30] for the continuity equation), when the coupling between both equations is present, the authors are aware only of references [16], for the scalar case  $d = 1$ , and [3], for the multidimensional case. However, in both references the structure of the system is *forward-forward*, i.e. both equations have initial conditions. This fact changes completely the theoretical and numerical analysis of the problem. As a matter of fact, for example in [3], the key property for convergence result of the proposed numerical scheme is a *one side Lipschitz condition* for  $Dv(\cdot, \cdot)$  of the form:

$$\exists C > 0 \text{ such that } \forall t \in [0, T], \quad \langle Dv(x, t) - Dv(y, t), x - y \rangle \geq -C|x - y|^2. \tag{1.3}$$

By the results in [27], condition (1.3) assures the stability of the so-called Fillipov characteristics and of the associated *measure solutions* of the continuity equation, which are the key to obtain their convergence result. Unfortunately, in our case (1.3) corresponds to the semiconvexity of  $v$ , which does not holds for an arbitrary time horizon  $T$  (see [11]).

Our line of research follows the ideas in [10], where a semi-discrete in time Semi-Lagrangian scheme is proposed to approximate (1.2) and a convergence result is obtained.

However, since the space variable is not discretized, the resulting scheme cannot be simulated. In this paper we propose a fully-discrete Semi-Lagrangian scheme for (1.2) and we study its main properties. We prove that the fully-discrete problem admits at least one solution and, for the case  $d = 1$ , we are able to prove the convergence of the scheme to a solution  $(v, m)$  of (1.2), when the discretization parameters tend to zero in a suitable manner. The key point of the proof is a weak semiconvavity property for the discretized solutions. Let us point out that our approximation scheme is presented in a general dimension  $d$  and several properties are proved in this generality. However, since in general (1.3) does not hold, uniform estimates in the  $L^\infty$  norm for the solutions of the scheme seems to be unavoidable in order to prove the convergence (see [12] for similar arguments regarding the vanishing viscosity approximation of (1.2)). Since we are able to prove these bounds only for  $d = 1$ , our convergence result for the fully-discrete scheme is valid only in this case.

The paper is organized as follows: In Section 2 we state our main assumptions, we collect some useful properties about semiconcave functions and we recall the main existence and uniqueness results for (1.2). In Section 3 we revisit the semi-discrete in time approximation of [10] and we improve some results, for example we prove uniform  $L^\infty$  bounds for the solutions of the semi-discrete scheme, which improves slightly the convergence result of [10]. Section 4 is devoted to the fully-discrete scheme. We establish the main properties of the scheme and we prove our main results: The fully-discrete scheme admits at least one solution and, if  $d = 1$  and the discretization parameters tend to zero in a suitable manner, every limit point of the solutions of the scheme is a solution of (1.2). Finally, in Section 5 we display some numerical simulations in the case of one space dimension.

## 2 Preliminaries

### 2.1 Basic assumptions and existence and uniqueness results for (1.2)

We denote by  $\mathcal{P}_1$  the set of the probability measures  $m$  such that  $\int_{\mathbb{R}^d} |x| dm(x) < \infty$ . The set  $\mathcal{P}_1$  is endowed with the Kantorovich-Rubinstein distance

$$\mathbf{d}_1(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} \phi(x) d[\mu - \nu](x) ; \phi : \mathbb{R}^d \rightarrow \mathbb{R} \text{ is 1-Lipschitz} \right\}. \quad (2.1)$$

Given a measure  $\mu \in \mathcal{P}_1$  we denote by  $\text{supp}(\mu)$  its support. In what follows, in order to simplify the notation, the operator  $D$  (resp.  $D^2$ ) will denote the derivative (resp. the second derivative) with respect to the space variable  $x \in \mathbb{R}^d$ . We suppose that the functions  $F, G : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}$  and the measure  $m_0$ , which are the data of (1.2), satisfy the following assumptions:

**(H1)**  $F$  and  $G$  are continuous over  $\mathbb{R}^d \times \mathcal{P}_1$ .

**(H2)** There exists a constant  $c_0 > 0$  such that for any  $m \in \mathcal{P}_1$

$$\|F(\cdot, m)\|_{C^2} + \|G(\cdot, m)\|_{C^2} \leq c_0,$$

where  $\|f(\cdot)\|_{C^2} := \sup_{x \in \mathbb{R}^d} \{|f(x)| + |Df(x)| + |D^2f(x)|\}$ .

**(H3)** The initial condition  $m_0 \in \mathcal{P}_1$  is absolutely continuous with respect to the Lebesgue measure, with density still denoted as  $m_0$ , and satisfies  $\text{supp}(m_0) \subset B(0, c_1)$  and  $\|m_0\|_\infty \leq c_1$ , for some  $c_1 > 0$ .

As a general rule in this paper, given an absolutely continuous measure (w.r.t the Lebesgue measure in  $\mathbb{R}^d$ )  $m \in \mathcal{P}_1$ , its density will still be denoted by  $m$ . Let us recall the definition of a solution  $(v, m)$  of (1.2) (see [21, 22]).

**Definition 2.1** *The pair  $(v, m) \in W_{loc}^{1,\infty}(\mathbb{R}^d \times [0, T]) \times L^1(\mathbb{R}^d \times (0, T))$  is a solution of (1.2) if the first equation is satisfied in the viscosity sense, while the second one is satisfied in the distributional sense. More precisely, for every  $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$*

$$\int_{\mathbb{R}^d} \phi(x, 0) m_0(x) dx + \int_0^T \int_{\mathbb{R}^d} [\partial_t \phi(x, t) - \langle Dv(x, t), D\phi(x, t) \rangle] m(x, t) dx dt = 0. \quad (2.2)$$

**Remark 2.1** *Classical arguments (see e.g. [5]) imply that (2.2) is equivalent to*

$$\int_{\mathbb{R}^d} \phi(x) m_0(x) dx - \int_0^t \int_{\mathbb{R}^d} \langle Dv(x, s), D\phi(x) \rangle m(x, s) dx ds = 0, \quad (2.3)$$

for all  $t \in [0, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ .

The following existence result is proved in [24, 12].

**Theorem 2.1** *Under (H1)-(H3) there exists at least a solution  $(v, m)$  of (1.2).*

A uniqueness result can be obtained assuming

**(H4)** The following *monotonicity* conditions hold true

$$\begin{aligned} \int_{\mathbb{R}^d} [F(x, m_1) - F(x, m_2)] d[m_1 - m_2](x) &\geq 0 \quad \text{for all } m_1, m_2 \in \mathcal{P}_1 \\ \int_{\mathbb{R}^d} [G(x, m_1) - G(x, m_2)] d[m_1 - m_2](x) &\geq 0 \quad \text{for all } m_1, m_2 \in \mathcal{P}_1. \end{aligned} \quad (2.4)$$

We have (see [24, 12]):

**Theorem 2.2** *Under (H1)-(H4) system (1.2) admits a unique solution  $(v, m)$ .*

## 2.2 Standard semiconcavity results

In the proof of Theorem 2.1, as well as in the the proof of our main results, the concept of semiconcavity plays a crucial role. For a complete account of the theory and its applications to the solution of HJB equations, we refer the reader to the book [11].

**Definition 2.2** *We say that  $w : \mathbb{R}^d \rightarrow \mathbb{R}$  is semiconcave with constant  $C_{conc} > 0$  if for every  $x_1, x_2 \in \mathbb{R}^d$ ,  $\lambda \in (0, 1)$  we have*

$$w(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda w(x_1) + (1 - \lambda)w(x_2) - \lambda(1 - \lambda) \frac{C_{conc}}{2} |x_1 - x_2|^2. \quad (2.5)$$

A function  $w$  is said to be semiconvex if  $-w$  is semiconcave.

Recall that for  $w : \mathbb{R}^d \rightarrow \mathbb{R}$  the super-differential  $D^+w(x)$  at  $x \in \mathbb{R}^d$  is defined as

$$D^+w(x) := \left\{ p \in \mathbb{R}^d ; \limsup_{y \rightarrow x} \frac{w(y) - w(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}. \quad (2.6)$$

We collect in the following Lemmas some useful properties of semiconcave functions (see [11]).

**Lemma 2.1** *For a function  $w : \mathbb{R}^d \rightarrow \mathbb{R}$ , the following assertions are equivalent:*

- (i) *The function  $w$  is semiconcave, with constant  $C_{conc}$ .*
- (ii) *For all  $x, y \in \mathbb{R}^d$ , we have*

$$w(x + y) + w(x - y) - 2w(x) \leq C_{conc}|y|^2.$$

- (iii) *For all  $x, y \in \mathbb{R}^d$  and  $p \in D^+w(x)$ ,  $q \in D^+w(y)$*

$$\langle q - p, y - x \rangle \leq C_{conc}|x - y|^2. \quad (2.7)$$

- (iv) *Setting  $I_d$  for the identity matrix, we have that  $D^2w \leq C_{conc}I_d$  in the sense of distributions.*

**Lemma 2.2** *Let  $w : \mathbb{R}^d \rightarrow \mathbb{R}$  be semiconcave. Then:*

- (i)  *$w$  is locally Lipschitz.*
- (ii) *If  $w_n$  is a sequence of semiconcave functions (with the same semiconcavity constant) converging point-wisely to  $w$ , then the convergence is locally uniform and  $Dw_n(\cdot) \rightarrow Dw(\cdot)$  a.e. in  $\mathbb{R}^d$ .*

### 2.3 Representation formulas for the solutions of the HJB and the continuity equations

Let  $\mu \in C([0, T]; \mathcal{P}_1)$  be given and let us denote by  $v[\mu]$  for the unique viscosity solution of

$$\left. \begin{aligned} -\partial_t v(x, t) + \frac{1}{2} |Dv(x, t)|^2 &= F(x, \mu(t)), \text{ in } \mathbb{R}^d \times (0, T), \\ v(x, T) &= G(x, \mu(T)) \text{ in } \mathbb{R}^d. \end{aligned} \right\} \quad (2.8)$$

Under assumptions **(H1)**-**(H2)**, standard results (see e.g. [8]) yield that for each  $(x, t) \in \mathbb{R}^d \times [0, T]$ , the following representation formula for  $v[\mu](x, t)$  holds true

$$\left. \begin{aligned} v[\mu](x, t) &= \inf_{\alpha \in L^2([t, T]; \mathbb{R}^d)} \left\{ \int_t^T \left[ \frac{1}{2} |\alpha(s)|^2 + F(X^{x,t}[\alpha](s), \mu(s)) \right] ds \right. \\ &\quad \left. + G(X^{x,t}[\alpha](T), \mu(T)), \right. \\ \text{where } X^{x,t}[\alpha](s) &:= x - \int_t^s \alpha(r) dr \quad \text{for all } s \in [t, T]. \end{aligned} \right\} \quad (\mathcal{CP})^{x,t}[\mu]$$

We set  $\mathcal{A}^{x,t}[\mu]$  for the set of *optimal controls*  $\alpha$  of  $(\mathcal{CP})^{x,t}[\mu]$ , i.e. for the set of solutions of  $(\mathcal{CP})^{x,t}[\mu]$ . Classical arguments imply that for all  $(x, t)$  the set  $\mathcal{A}^{x,t}[\mu]$  is non empty.

We now collect some important well known properties of problem  $(\mathcal{CP})^{x,t}[\mu]$  (see e.g. [11, 12]).

**Proposition 2.1** *Under **(H1)**-**(H2)**, The value function  $v[\mu]$  satisfies the following properties:*

- (i) *We have that  $(x, t) \rightarrow v[\mu](x, t)$  is Lipschitz, with a Lipschitz constant independent of  $\mu$ .*
- (ii) *For all  $t \in [0, T]$  the function  $v[\mu](\cdot, t) \in \mathbb{R}$  is semiconcave, uniformly with respect to  $\mu$ .*
- (iii) *There exists a constant  $c_2 > 0$  (independent of  $(\mu, x, t)$ ) such that*

$$\|\alpha\|_{L^\infty([t, T]; \mathbb{R}^d)} \leq c_2 \quad \text{for all } \alpha \in \mathcal{A}^{x,t}[\mu].$$

- (iv) *For all  $(x, t)$  and  $\alpha \in \mathcal{A}^{x,t}[\mu]$ , we have that*

$$\alpha(t) \in D^+ v[\mu](x, t). \quad (2.9)$$

- (v) *For all  $t \in [0, T]$  the function  $v[\mu](\cdot, t)$  is differentiable at  $x$  iff there exists  $\alpha \in \mathcal{A}^{x,t}[\mu]$  such that  $\mathcal{A}^{x,t}[\mu] = \{\alpha\}$ . In this case, we have that*

$$Dv[\mu](x, t) = \alpha(t). \quad (2.10)$$

- (vi) *For every  $s \in (t, T]$  and  $\alpha \in \mathcal{A}^{x,t}[\mu]$ , we have that  $v[\mu](\cdot, \cdot)$  is differentiable at  $(X^{x,t}[\alpha](s), s)$ .*

Now, we define a *measurable selection of optimal flows*, i.e. of optimal trajectories for the family of problems  $\{(\mathcal{CP})^{x,t}[\mu] ; (x, t) \in \mathbb{R}^d \times [0, T]\}$ . Classical arguments (see [12, 6]) show that the multivalued map  $(x, t) \rightarrow \mathcal{A}^{x,t}[\mu]$ , admits a measurable selection  $\alpha^{x,t}[\mu](\cdot)$ . Given  $(x, t)$  the *flow*  $\Phi[\mu](x, t, \cdot)$  is defined as

$$\Phi[\mu](x, t, s) := x - \int_t^s \alpha^{x,t}[\mu](r) dr \quad \text{for all } s \in [t, T]. \quad (2.11)$$

By Proposition 2.1(v)-(vi), omitting the dependence on  $\mu$  for notational convenience,  $\Phi(x, t, \cdot)$  satisfies

$$\left. \begin{aligned} \frac{\partial}{\partial s} \Phi(x, t, s) &= -Dv[\mu](\Phi(x, t, s), s) \quad \text{for } s \in (t, T), \\ \Phi(x, t, t) &= x. \end{aligned} \right\} \quad (2.12)$$

For all  $t \in [0, T]$ , let us define  $m[\mu](t)$  as the initial measure  $m_0$  transported by the flow  $\Phi[\mu]$ . More precisely,

$$m[\mu](t) := \Phi[\mu](\cdot, 0, t) \# m_0, \quad (2.13)$$

i.e.

$$m[\mu](t)(A) = m_0(\Phi[\mu]^{-1}(\cdot, 0, t)(A)) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d),$$

or equivalently, for all bounded and continuous  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\int_{\mathbb{R}^d} \phi(x) d[m[\mu](t)](x) = \int_{\mathbb{R}^d} \phi(\Phi[\mu](x, 0, t)) dm_0(x).$$

Since  $\Phi[\mu](x, \cdot, \cdot)$  satisfies the *semigroup property*, omitting the dependence on  $\mu$  for simplicity,

$$\Phi(x, s, t) = \Phi(\Phi(x, s, r), r, t) \quad \text{for all } r \in [s, t],$$

we easily check that

$$m[\mu](t) := \Phi(\cdot, r, t) \# [\Phi(\cdot, 0, r) \# m_0] = \Phi(\cdot, r, t) \# [m[\mu](r)] \quad \text{for all } r \in [s, t]. \quad (2.14)$$

The fundamental result is the following

**Proposition 2.2** *There exists a constant  $c_3 > 0$  (independent of  $(\mu, x, y, r, t)$ ), such that*

$$|\Phi(x, r, t) - \Phi(y, r, t)| \geq c_3 |x - y| \quad \text{for all } 0 \leq r \leq t, \quad \text{and } x, y \in \mathbb{R}^d.$$

The key in the proof of the above Proposition (see e.g. [12, Lemma 4.13]) is the semiconcavity of  $v[\mu](\cdot, t)$ , which is uniform w.r.t  $\mu$ , and Gronwall's Lemma. As a consequence we have that (see e.g. [12, Theorem 4.18 and Lemma 4.14]).

**Theorem 2.3** *We have that  $m[\mu](\cdot)$  is the unique solution (in the distributional sense) of*

$$\left. \begin{aligned} \partial_t m(x, t) - \operatorname{div}(Dv[\mu](x, t)m(x, t)) &= 0, \quad \text{in } \mathbb{R}^d \times (0, T), \\ m(x, 0) &= m_0(x) \quad \text{in } \mathbb{R}^d. \end{aligned} \right\} \quad (2.15)$$

Moreover, there exists a constant  $c_4 > 0$ , independent of  $\mu$ , such that  $m[\mu]$  satisfies the following properties:

- (i) For all  $t \in [0, T]$ , the measure  $m[\mu](t)$  is absolutely continuous (with density still denoted by  $m[\mu](t)$ ), has a support in  $B(0, c_4)$  and  $\|m[\mu](t)\|_\infty \leq c_4$ .
- (ii) For all  $t, t' \in [0, T]$ , we have that

$$\mathbf{d}_1(\mu(t), \mu(t')) \leq c_4 |t - t'|.$$

**Remark 2.2** *In the proof of the above result (see [12]) Proposition 2.2 is crucial in order to show that the transported measure  $m[\mu](\cdot)$  is absolutely continuous, as  $m_0$ , and its density remains uniformly bounded in  $L^\infty(\mathbb{R}^d)$ .*

Theorem 2.3 (i)-(ii) implies that  $m[\mu](\cdot) \in C([0, T]; \mathcal{P}_1)$ . We thus see that (1.2) is equivalent to find  $m \in C([0, T]; \mathcal{P}_1)$ , such that

$$m(t) = \Phi[m](\cdot, 0, t) \# m_0 \quad \text{for all } t \in [0, T]. \quad (\text{MFG})$$

### 3 A revisit to the semi-discrete in time approximation

In this section we review the semi-discrete in time approximation studied in [10] and we improve some results.

#### 3.1 Semi-discretization of the HJB equation

Given  $h > 0$  and  $N \in \mathbb{N}$  such that  $Nh = T$ , we set  $t_k := kh$  for  $k = 0, \dots, N$ . Let us define the following spaces:

$$\begin{aligned} \mathcal{K}_N &:= \left\{ \mu = (\mu_\ell)_{\ell=0}^N : \text{ such that } \mu_\ell \in \mathcal{P}_1 \text{ for all } \ell = 0, \dots, N \right\}, \\ \mathcal{A}_k &:= \left\{ \alpha = (\alpha_\ell)_{\ell=k}^{N-1} : \text{ such that } \alpha_\ell \in \mathbb{R}^d \right\} \quad \text{for } k = 0, \dots, N-1. \end{aligned}$$

For  $\mu \in \mathcal{K}_N$  and  $k = 1, \dots, N$ , we consider the following semi-discrete approximation of  $(\mathcal{CP})^{x,t}[\mu]$

$$\left. \begin{aligned} v_k[\mu](x) &:= \inf_{\alpha \in \mathcal{A}_k} \left\{ \sum_{\ell=k}^{N-1} \left[ \frac{1}{2} |\alpha_\ell|^2 + F(X_\ell^{x,k}[\alpha], \mu_\ell) \right] h + G(X_N^{x,k}[\alpha], \mu_N) \right\}, \\ \text{where } X_{\ell+1}^{x,k}[\alpha] &:= X_\ell^{x,k}[\alpha] - h\alpha_\ell \quad \text{for } \ell = k, \dots, N-1, \\ X_k^{x,k}[\alpha] &:= x \end{aligned} \right\} \quad (\mathcal{CP})_h^{x,k}[\mu].$$



Note that no discretization is performed in the space variable. As for the continuous time problem, we have that  $(\mathcal{CP})_h^{x,k}[\mu]$  admits at least a solution for all  $(x, k)$ . We set by  $\mathcal{A}_k[\mu](x) \subseteq \mathcal{A}_k$  for the set of optimal solutions of  $(\mathcal{CP})_h^{x,k}[\mu]$ , i.e. the set of *discrete optimal controls*. By the discrete dynamic programming principle (see e.g. [8]),  $v_k[\mu](\cdot)$  can be recursively calculated as

$$\left. \begin{aligned} v_k[\mu](x) &= \inf_{\alpha \in \mathbb{R}^d} \left\{ v_{k+1}[\mu](x - h\alpha) + \frac{1}{2}h|\alpha|^2 \right\} + hF(x, \mu_k), \quad k = 0, \dots, N-1, \\ v_N[\mu](x) &= G(x, \mu_N), \end{aligned} \right\} \quad (3.1)$$

which is a “semi-discrete in time version” of (2.8). Let us set,

$$v_h[\mu](x, t) := v_{[t/h]}[\mu](x) \quad \text{for all } t \in [0, T].$$

for the classical “extension” of  $v_k[\mu](\cdot)$  to a function defined on  $\mathbb{R}^d \times [0, T]$ . Now, we provide the “discrete” analogous results to those of Proposition 2.1.

**Proposition 3.1** *For all  $h > 0$ , we have:*

- (i) *For any  $t \in [0, T]$ , the function  $v_h[\mu](\cdot, t)$  is Lipschitz continuous, with a Lipschitz constant independent of  $\mu$ .*
- (ii) *For all  $t \in [0, T]$  the function  $v_h[\mu](\cdot, t)$  is semiconcave, uniformly in  $(h, \mu, t)$ .*
- (iii) *There exists a constant  $c_5 > 0$  (independent of  $(\mu, h, x, k)$ ) such that*

$$\max_{\ell=k, \dots, N-1} |\alpha_\ell| \leq c_5 \quad \text{for all } \alpha \in \mathcal{A}_k[\mu](x).$$

- (iv) *For all  $x \in \mathbb{R}^d$ ,  $k = 0, \dots, N-1$  and  $\alpha \in \mathcal{A}_k[\mu](x)$ , we have*

$$\alpha_\ell + hDF \left( X_\ell^{x,k}[\alpha], \mu_\ell \right) \in D^+ v_h[\mu] \left( X_\ell^{x,k}[\alpha], t_\ell \right) \quad \text{for } \ell = k, \dots, N-1.$$

- (v) *We have that  $v_h[\mu](\cdot, t)$  is differentiable at  $x$  iff for  $k = [t/h]$  there exists  $\alpha \in \mathcal{A}_k$  such that  $\mathcal{A}_k[\mu](x) = \{\alpha\}$ . In that case, the following holds:*

$$Dv_h[\mu](x, t) = \alpha_k + hDF(x, \mu_k).$$

- (vi) *Given  $(x, t)$  and  $\alpha \in \mathcal{A}_k[\mu](x)$ , with  $k = [t/h]$ , we have that for all  $s \in [t_{k+1}, T]$ , the function  $v_h[\mu](\cdot, s)$  is differentiable at  $X_\ell^{x,k}[\alpha]$ , with  $\ell = [s/h]$ .*

**Proof.** We only prove (iv) since the other statements are proved in [10]. For notational convenience, we omit the  $\mu$  argument and we prove the result for  $\ell = k$ , since for  $\ell = k+1, \dots, N$  the assertion follows from (v)-(vi). Let  $x, y \in \mathbb{R}^d$  and  $\sigma \geq 0$ . Since  $\alpha \in \mathcal{A}_k[\mu](x)$ , we have

$$v_k(x + \sigma y) \leq \sum_{\ell=k}^{N-1} \left[ \frac{1}{2}|\alpha_\ell|^2 + F \left( X_\ell^{x+\sigma y, k}[\alpha], \mu_\ell \right) \right] h + G \left( X_N^{x+\sigma y, k}[\alpha], \mu_N \right),$$

with equality for  $\sigma = 0$ . Therefore,

$$\begin{aligned} v_k(x + \sigma y) - v_k(x) &\leq h \sum_{\ell=k}^{N-1} \left[ F \left( X_\ell^{x+\sigma y, k}[\alpha], \mu_\ell \right) - F \left( X_\ell^{x, k}[\alpha], \mu_\ell \right) \right] \\ &\quad + G \left( X_N^{x+\sigma y, k}[\alpha], \mu_N \right) - G \left( X_\ell^{x, k}[\alpha], \mu_N \right). \end{aligned} \quad (3.2)$$

On the other hand, the optimality condition for  $\alpha$  yields

$$\alpha_k = h \sum_{\ell=k+1}^{N-1} DF \left( X_\ell^{x, k}[\alpha], \mu_\ell \right) + DG \left( X_\ell^{x, k}[\alpha], \mu_N \right).$$

Combining with (3.2) and taking the limit as  $\sigma \rightarrow 0$ , gives

$$\limsup_{\sigma \rightarrow 0} \frac{v_k(x + \sigma y) - v_k(x)}{\sigma} - \langle \alpha_k + hDF(x, \mu_k), y \rangle \leq 0,$$

which, by [11, Proposition 3.15 and Theorem 3.2.1], implies the result. ■

Given  $(x, k)$  and  $\alpha \in \mathcal{A}_k[\mu](x)$  we set

$$\alpha_k[\mu](x) := \alpha_k. \quad (3.3)$$

Proposition 3.1(iv) implies that

$$\alpha_k[\mu](x) \in D^+ v_h[\mu](x, t_k) - hDF(x, u_k). \quad (3.4)$$

A straightforward computation shows that  $\alpha_k[\mu](x)$  solves, for each  $(x, k)$ , the problem defined in (3.1). Moreover, by Proposition 3.1(v)-(vi), the following relation holds true

$$\alpha_\ell = \alpha_\ell[\mu] \left( X_\ell^{x, k}[\alpha] \right) \quad \text{for all } \ell = k, \dots, N-1. \quad (3.5)$$

### 3.2 Semi-discretization of the continuity equation

Let  $\alpha^{x, k}[\mu] \in \mathcal{A}_k$  be a measurable selection of the multifunction  $(x, k) \rightarrow \mathcal{A}_k[\mu](x)$ . Given this measurable selection, we set  $\alpha_k[\mu](x) = \alpha_k^{x, k}[\mu]$ , as in (3.3). By (3.4) - (3.5), there exists a measurable function  $(x, k) \rightarrow p_k[\mu](x) \in \mathbb{R}^d$  such that  $p_k[\mu](x) \in D^+ v_k[\mu](x)$  and for all time iterations  $\ell = k, \dots, N$  we have

$$\alpha_\ell[\mu] \left( X_\ell^{x, k}[\alpha^{x, k}[\mu]] \right) = p_\ell[\mu] \left( X_\ell^{x, k}[\alpha^{x, k}[\mu]] \right) - hDF \left( X_\ell^{x, k}[\alpha^{x, k}[\mu]], \mu_\ell \right). \quad (3.6)$$

Moreover, Proposition 3.1(v)-(vi) implies that for  $\ell = k+1, \dots, N$

$$p_\ell[\mu] \left( X_\ell^{x, k}[\alpha_\ell^{x, k}[\mu]] \right) = Dv_\ell[\mu] \left( X_\ell^{x, k}[\alpha_\ell^{x, k}[\mu]] \right) \quad \text{for all } x \in \mathbb{R}^d \quad (3.7)$$

and

$$p_k[\mu](x) = Dv_k[\mu](x) \quad \text{for a.a. } x \in \mathbb{R}^d. \quad (3.8)$$

Given  $(x, k_1)$ , the *discrete flow*  $\Phi_{k_1, \cdot}[\mu](x) \in \mathbb{R}^{(N-k) \times d}$  is defined as

$$\Phi_{k_1, k_2}[\mu](x) := x - h \sum_{\ell=k_1}^{k_2-1} \alpha_\ell^{x, k_1}[\mu] \quad \text{for all } k_2 \geq k_1. \quad (3.9)$$

Equivalently, by (3.6), for all  $k_1 \leq k_2 \leq k_3$ ,

$$\begin{aligned} \Phi_{k_1, k_3}[\mu](x) &:= x - h \sum_{\ell=k_1}^{k_3-1} \alpha_\ell[\mu] \left( X_\ell^{x, k_1}[\alpha^{x, k_1}[\mu]] \right), \\ &= \Phi_{k_1, k_2}[\mu](x) - h \sum_{\ell=k_2}^{k_3-1} \alpha_\ell[\mu] \left( X_\ell^{x, k_1}[\alpha^{x, k_1}[\mu]] \right). \end{aligned} \quad (3.10)$$

In particular, for all  $k_1 \leq k_2$ ,

$$\Phi_{k_1, k_2+1}[\mu](x) = \Phi_{k_1, k_2}[\mu](x) - h \alpha_{k_2}[\mu] \left( \Phi_{k_1, k_2}[\mu](x) \right). \quad (3.11)$$

The following result, analogous to Proposition 2.2, is an important improvement of [10, Lemma 3.6].

**Proposition 3.2** *There exists a constant  $c_6 > 0$  (independent of  $\mu$  and small enough  $h$ ) such that for all  $k = 1, \dots, N$  and  $x, y \in \mathbb{R}^d$  we have*

$$|\Phi_{0, k}[\mu](x) - \Phi_{0, k}[\mu](y)| \geq c_6 |x - y|. \quad (3.12)$$

*Thus,  $\Phi_{0, k}[\mu](\cdot)$  is invertible in  $\Phi_{0, k}[\mu](\mathbb{R}^d)$  and the inverse  $\Upsilon_{0, k}[\mu](\cdot)$  is  $1/c_6$ -Lipschitz.*

**Proof.** For notational convenience, let us set  $\Phi_k = \Phi_{0, k}[\mu](x)$  and  $\Psi_k = \Phi_{0, k}[\mu](y)$ . Expression (3.11) implies that

$$|\Phi_{k+1} - \Psi_{k+1}|^2 \geq |\Phi_k - \Psi_k|^2 - 2h [\alpha_k[\mu](\Phi_k) - \alpha_k[\mu](\Psi_k)] \cdot (\Phi_k - \Psi_k). \quad (3.13)$$

By (3.6) we have (omitting the dependence on  $\mu$ )

$$\alpha_k(\Phi_k) - \alpha_k(\Psi_k) = p_k(\Phi_k) - p_k(\Psi_k) - h [DF(\Phi_k) - DF(\Psi_k)].$$

By the semiconcavity of  $v_k[\mu](\cdot)$  and the semiconvexity of  $F(\cdot, \mu(t_l))$ , Lemma 2.1(iii) gives

$$[\alpha_k(\Phi_k) - \alpha_k(\Psi_k)] \cdot (\Phi_k - \Psi_k) \leq c(1 + h) |\Phi_k - \Psi_k|^2, \quad (3.14)$$

for some  $c > 0$ . By (3.13) and (3.14), there is  $c' > 0$  (independent of  $h$  small enough) such that

$$|\Phi_{k+1} - \Psi_{k+1}|^2 \geq (1 - hc') |\Phi_k - \Psi_k|^2.$$

Therefore, for every  $k = 1, \dots, N$ , we get

$$|\Phi_{k+1} - \Psi_{k+1}|^2 \geq (1 - hc')^k |x - y|^2 \geq (1 - hc')^{[T/h]} |x - y|^2.$$

and the result follows from the convergence of  $(1 - hc')^{[T/h]}$  to  $\exp(-c'T)$  as  $h \downarrow 0$ . ■

A natural semi-discretization of the solution  $m[\mu]$  of (2.15), whose representation formula is given by (2.13), is then obtained as the push-forward of  $m_0$  under the discrete flow  $\Phi_{0,k}[\mu](\cdot)$ , i.e. for every  $k = 0, \dots, N$  we define

$$m_k[\mu] := \Phi_{0,k}[\mu](\cdot) \# m_0. \quad (3.15)$$

By (3.10) we have

$$m_k[\mu] = \Phi_{\ell,k}[\mu](\cdot) \# m_\ell[\mu] \quad \text{for all } \ell = 1, \dots, k, \quad (3.16)$$

which is the analogous to (2.14), for the continuous time case. In particular, for all  $\phi \in C_b(\mathbb{R}^d)$  (space of bounded and continuous functions over  $\mathbb{R}^d$ ), we have

$$\int_{\mathbb{R}^d} \phi(x) dm_{k+1}[\mu](x) = \int_{\mathbb{R}^d} \phi(x - h\alpha_k[\mu](x)) dm_k[\mu](x), \quad (3.17)$$

which applied with  $\phi \equiv 1$  gives  $m_k[\mu](\mathbb{R}^d) = 1$  for  $k = 0, \dots, N$ .

We have the following Lemma, which improves [10, Lemma 3.7] since we now prove, using Proposition 3.2, uniform bounds for the density of  $m_k[\mu]$ .

**Lemma 3.1** *There exist  $c_7 > 0$  (independent of  $(\mu, h)$ ) such that:*

(i) *For all  $k_1, k_2 \in \{1, \dots, N\}$ , we have that*

$$\mathbf{d}_1(m_{k_1}[\mu], m_{k_2}[\mu]) \leq c_7 h |k_1 - k_2| = c_7 |t_{k_1} - t_{k_2}|. \quad (3.18)$$

(ii) *For all  $k = 1, \dots, N$ ,  $m_k[\mu]$  is absolutely continuous (with density still denoted by  $m_k[\mu]$ ), has a support in  $B(0, c_7)$  and  $\|m_k[\mu]\|_\infty \leq c_7$ .*

**Proof.** By Proposition 3.1(iii) we have

$$|\Phi_{0,k_1}[\mu](x) - \Phi_{0,k_2}[\mu](x)| \leq c_5 h |k_1 - k_2| = c_5 |t_{k_1} - t_{k_2}|. \quad (3.19)$$

By definition of  $m_k[\mu](\cdot)$ , we have that for any 1-Lipschitz function  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x) d[m_{k_1}[\mu] - m_{k_2}[\mu]](x) &\leq \int_{\mathbb{R}^d} |\Phi_{0,k_1}[\mu](x) - \Phi_{0,k_2}[\mu](x)| dm_0(x) \\ &\leq c_5 h |k_1 - k_2| = c_5 |t_{k_1} - t_{k_2}|. \end{aligned}$$

On the other hand, since by **(H1)** we have  $\text{supp}(m_0) \subset B(0, c_1)$ , Proposition 3.1(iii) implies that  $\text{supp}(m_k[\mu])$  is contained in  $B(0, c_1 + 2c_5 T)$ . Moreover, for any Borel set  $A$  and  $k = 1, \dots, N$ , Proposition 3.2 and the fact that  $\|m_0\|_\infty \leq c_1$  imply

$$m_k[\mu](A) = m_0(\Upsilon_{0,k}[\mu](A)) \leq \|m_0\|_\infty |\Upsilon_{0,k}(A)| \leq \frac{c_1}{c_6} |A|,$$

where  $|A|$  denotes the Lebesgue measure of the set  $A$ . Thus,  $m_k[\mu]$  is absolutely continuous and its density, still denoted by  $m_k[\mu]$ , satisfies  $\|m_k[\mu]\|_\infty \leq \frac{c_1}{c_6}$ . The result follows by taking  $c_7 = \max\{c_5, c_1 + 2c_5T, c_1/c_6\}$ . ■

We now define

$$m_h[\mu](t) := \frac{t_{k+1} - t}{h} m_k[\mu] + \frac{t - t_k}{h} m_{k+1}[\mu] \quad \text{if } t \in [t_k, t_{k+1}]. \quad (3.20)$$

The following result is a clear consequence of Lemma 3.1 and (3.20).

**Proposition 3.3** *There exists constants  $c_8 > 0$  (independent of  $\mu$  and small enough  $h$ ) such that:*

(i) *For all  $t_1, t_2 \in [0, T]$ , we have that*

$$\mathbf{d}_1(m_h[\mu](t_1), m_h[\mu](t_2)) \leq c_8 |t_1 - t_2|. \quad (3.21)$$

(ii) *For all  $t \in [0, T]$ ,  $m_h[\mu](t)$  is absolutely continuous (with density denoted by  $m_h[\mu](\cdot, t)$ ), has a support in  $B(0, c_8)$  and  $\|m_h[\mu](\cdot, t)\|_\infty \leq c_8$ .*

### 3.3 The semi-discrete scheme for the first order MFG problem (1.2)

For a given  $N > 0$ , consider the following *semi-discretization* of (MFG):

$$\text{Find } m \in \mathcal{K}_N \text{ such that } m_k = \Phi_{0,k}[m](\cdot) \# m_0 \quad \text{for all } k = 0, \dots, N, \quad (\text{MFG})_h.$$

The following result is proved in [10].

**Theorem 3.1** *Under (H1)-(H3) we have that  $(\text{MFG})_h$  admits at least one solution  $m_h \in \mathcal{K}_N$ . Moreover, if (H4) holds, the solution is unique.*

Given any solution  $m_h$  of  $(\text{MFG})_h$ , using (3.20) we define an element, still denoted by  $m_h$ , in  $C([0, T]; \mathcal{P}_1)$ .

**Theorem 3.2** *Under (H1)-(H3) every limit point of  $m_h$  in  $C([0, T]; \mathcal{P}_1)$ , as  $h \downarrow 0$ , solves (MFG). In particular, if (H4) holds we have that  $m_h \rightarrow m$  (the unique solution of (MFG)) in  $C([0, T]; \mathcal{P}_1)$  and in  $L^\infty(\mathbb{R}^d \times [0, T])$ -weak- $*$ .*

**Proof.** Proposition 3.3 and Ascoli Theorem imply that  $m_h$  has at least one limit point  $\bar{m}$  in  $C([0, T]; \mathcal{P}_1)$  as  $h \downarrow 0$ . The fact that  $\bar{m}$  solves (MFG) is proved in [10, Theorem 4.3] using optimal control techniques. Finally, by Proposition 3.3(ii) we have that  $m_h \rightarrow \bar{m}$  in  $L^\infty(\mathbb{R}^d \times [0, T])$ -weak- $*$ . The result follows. ■

## 4 The fully-discrete scheme

Given  $h, \rho > 0$ , we consider a  $d$  dimensional lattice  $\mathcal{G}_\rho := \{x_i = i\rho, i \in \mathbb{Z}^d\}$  and a time-space grid  $\mathcal{G}_{\rho,h} := \mathcal{G}_\rho \times \{t_k\}_{k=0}^N$ , where  $t_k = kh$  ( $k = 0, \dots, N$ ) and  $t_N = Nh = T$ . We set  $B(\mathcal{G}_\rho)$  and  $B(\mathcal{G}_{\rho,h})$  for the space of bounded functions defined on  $\mathcal{G}_\rho$  and  $\mathcal{G}_{\rho,h}$ , respectively. Given  $f \in B(\mathcal{G}_\rho)$  and  $g \in B(\mathcal{G}_{\rho,h})$  we will use the notation

$$f_i := f(x_i), \quad g_{i,k} := g(x_i, t_k) \quad \text{for all } i \in \mathbb{Z}^d \text{ and } k = 0, \dots, N.$$

Let us consider the  $\mathbb{P}_1$  basis  $\{\beta_i ; i \in \mathbb{Z}^d\}$ , where the function  $\beta_i : \mathbb{R}^d \rightarrow \mathbb{R}$  is defined by  $\beta_i(x) := [1 - \frac{\|x - x_i\|_1}{\rho}]_+ := \max\{[1 - \frac{\|x - x_i\|_1}{\rho}], 0\}$ . Denoting by  $e_1, \dots, e_d$  the canonical base of  $\mathbb{R}^d$ , it is easy to verify that  $\beta_i(x)$  is continuous with compact support contained in  $Q(x_i) := [x_i - \rho e_1, x_i + \rho e_1] \times \dots \times [x_i - \rho e_d, x_i + \rho e_d]$ ,  $0 \leq \beta_i \leq 1$ ,  $\beta_i(x_j) = \delta_{ij}$  (the Kronecker symbol) and  $\sum_{i \in \mathbb{Z}^d} \beta_i(x) = 1$ . Let us consider the *interpolation operator*  $I[\cdot] : B(\mathcal{G}_\rho) \rightarrow C_b(\mathbb{R}^d)$ , as

$$I[f](\cdot) := \sum_{i \in \mathbb{Z}^d} f_i \beta_i(\cdot). \quad (4.1)$$

We recall a standard estimate for  $I$  (see e.g. [14, 28]). Given  $\phi \in C_b(\mathbb{R}^d)$ , let us define  $\hat{\phi} \in B(\mathcal{G}_\rho)$  by  $\hat{\phi}_i := \phi(x_i)$  for all  $i \in \mathbb{Z}^d$ . We have that

$$\sup_{x \in \mathbb{R}^d} |I[\hat{\phi}](x) - \phi(x)| = O(\rho^\gamma), \quad (4.2)$$

where  $\gamma = 1$  if  $\phi$  is Lipschitz and  $\gamma = 2$  if  $\phi \in C^2(\mathbb{R}^d)$ .

### 4.1 The fully-discrete scheme for the HJB equation

For a given  $\mu \in C([0, T], \mathcal{P}_1)$ , we define recursively  $v \in B(\mathcal{G}_{\rho,h})$  using the following *Semi-Lagrangian scheme* for (2.8):

$$v_{i,k} = S_{\rho,h}[\mu](v_{\cdot,k+1}, i, k) \quad k = 0, \dots, N-1 \quad \text{and} \quad v_{i,N} = G(x_i, \mu(t_N)), \quad (4.3)$$

where  $S_{\rho,h}[\mu] : B(\mathcal{G}_\rho) \times \mathbb{Z}^d \times \{0, \dots, N-1\} \rightarrow \mathbb{R}$  is defined as

$$S_{\rho,h}[\mu](f, i, k) := \inf_{\alpha \in \mathbb{R}^d} [I[f](x_i - h\alpha) + \frac{1}{2}h|\alpha|^2] + hF(x_i, \mu(t_k)). \quad (4.4)$$

The following properties of  $S_{\rho,h}[\mu]$  are a straightforward consequence of the definition and assumptions **(H1)** and **(H2)**.

**Lemma 4.1** *The following assertions hold true:*

(i) [The scheme is well defined] *There exists at least one  $\alpha \in \mathbb{R}^d$  that minimizes the r.h.s. of (4.4). Moreover, there exists  $c_9 > 0$  such that  $\sup_{i \in \mathbb{Z}^d, k=0, \dots, N} |v_{i,k}| \leq c_9$ .*

(ii)[Monotonicity] For all  $v, w \in B(\mathcal{G}_\rho)$  with  $v \leq w$ , we have that

$$S_{\rho,h}[\mu](v, i, k) \leq S_{\rho,h}[\mu](w, i, k) \quad \forall i \in \mathbb{Z}^d, k = 0, \dots, N. \quad (4.5)$$

(iii) For every  $K \in \mathbb{R}$  and  $w \in B(\mathcal{G}_\rho)$  we have

$$S_{\rho,h}[\mu](w + K, i, n) = S_{\rho,h}[\mu](w, i, n) + K. \quad (4.6)$$

(iv)[Consistency] Let  $(\rho_n, h_n) \rightarrow 0$  (as  $n \uparrow \infty$ ) and consider a sequence of grid points  $(x_{i_n}, t_{k_n}) \rightarrow (x, t)$  and a sequence  $\mu_n \in C([0, T]; \mathcal{P}_1)$  such that  $\mu_n \rightarrow \mu$ . Then, for every  $\phi \in C^1(\mathbb{R}^d \times [0, T])$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} [\phi(x_{i_n}, t_{k_n}) - S_{\rho_n, h_n}[\mu_n](\phi_{k_{n+1}}, i_n, k_n)] = -\partial_t \phi(x, t) + \frac{1}{2} |D\phi(x, t)(x, t)|^2 - F(x, \mu(t)). \quad (4.7)$$

where  $\phi_k = \{\phi(x_i, t_k)\}_{i \in \mathbb{Z}^d}$ .

We define

$$v_{\rho,h}[\mu](x, t) := I[v, \frac{t}{h}](x) \quad \text{for all } (x, t) \in \mathbb{R}^d \times [0, T], \quad (4.8)$$

where we recall that  $v_{i,k}$  is defined by (4.3).

The following notion of *weak semiconcavity* (see e.g. [23]), will be very useful.

**Definition 4.1** Given  $C, \rho > 0$ , we say that  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $(C, \rho)$ -weakly semiconcave if for all  $x, y \in \mathbb{R}^d$  and  $\lambda \in [0, 1]$  we have

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y) + \frac{C}{2} \lambda(1 - \lambda) (|x - y|^2 + \rho^2). \quad (4.9)$$

For the sake of completeness, we recall the following elementary properties of weak semiconcave whose proofs are easy adaptations of the semiconcave case.

**Lemma 4.2** For a continuously differentiable function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  the following assertions are equivalent:

(i) The function  $f$  is  $(C, \rho)$ -weakly semiconcave.

(ii) For every  $x, y \in \mathbb{R}^d$

$$f(x + y) - 2f(x) + f(x - y) \leq C(|y|^2 + \rho^2). \quad (4.10)$$

(iii) For every  $x, y \in \mathbb{R}^d$

$$f(y) - f(x) - \langle Df(x), y - x \rangle \leq \frac{C}{2} (|y - x|^2 + \rho^2). \quad (4.11)$$

In particular, if  $f$  is  $(C, \rho)$ -weakly semiconcave then

$$\langle Df(y) - Df(x), y - x \rangle \leq C (|y - x|^2 + \rho^2) \quad \forall x, y \in \mathbb{R}^d. \quad (4.12)$$

The following result yields the weak semiconcavity of  $v_{\rho,h}[\mu]$ .

**Lemma 4.3** *For every  $t \in [0, T]$ , the following assertions hold true:*

- (i) [Lipschitz property] *The function  $v_{\rho,h}[\mu](\cdot, t)$  is Lipschitz with constant independent of  $(\rho, h, \mu, t)$ .*
- (ii) [Weak semiconcavity] *The function  $v_{\rho,h}[\mu](\cdot, t)$  is  $(C, \rho)$ -weakly semiconcave, with  $C$  independent of  $(\rho, h, \mu, t)$ .*

**Proof.** By **(H2)** we have that  $\|DG(\cdot, \mu(T))\|_\infty \leq c_0$  and so  $I[G](\cdot, \mu(T))$  is  $c_0$ -Lipschitz. Thus, by the (4.3) and (4.8), we get that  $v_{\rho,h}[\mu](\cdot, t_{N-1})$  is Lipschitz with constant  $hc_0 + c_0$ . Iterating the argument, using **(H2)** for  $F$ , we get that  $v_{\rho,h}[\mu](\cdot, t)$  is  $c_0(1 + T)$  Lipschitz for all  $t \in [0, T]$ . The proof of the second assertion is provided e.g. in [3, Lemma 4.1]. ■

**Theorem 4.1** *Let  $(\rho_n, h_n) \rightarrow 0$  (as  $n \uparrow \infty$ ) be such that  $\frac{\rho_n^2}{h_n} \rightarrow 0$ . Then, for every sequence  $\mu_n \in C([0, T]; \mathcal{P}_1)$  such that  $\mu_n \rightarrow \mu$  in  $C([0, T]; \mathcal{P}_1)$ , we have that  $v_{\rho_n, h_n}[\mu_n] \rightarrow v[\mu]$  uniformly over compact sets.*

**Proof.** Using assumption **(H1)**, the proof of this results is a straightforward variation of the proof in [13], which is a revised proof of the result given in [9]. However, for the sake of completeness we provide the details. For  $(y, s) \in \mathbb{R}^d \times [0, T]$ , set

$$v^*(y, s) := \limsup_{\substack{(y', s') \rightarrow (y, s) \\ n \rightarrow \infty}} v_{\rho_n, h_n}[\mu_n](y', s'), \quad v_*(y, s) := \liminf_{\substack{(y', s') \rightarrow (y, s) \\ n \rightarrow \infty}} v_{\rho_n, h_n}[\mu_n](y', s').$$

Let us prove that  $v^*$  is a viscosity subsolution of

$$\begin{aligned} -\partial_t v(x, t) + \frac{1}{2}|Dv(x, t)|^2 &= F(x, \mu(t)) \quad \text{for } (x, t) \in \mathbb{R}^d \times (0, T), \\ v(x, T) &= G(x, \mu(T)) \quad \text{for } x \in \mathbb{R}^d. \end{aligned} \tag{4.13}$$

Let  $(\bar{y}, \bar{s}) \in \mathbb{R}^d \times (0, T)$  and  $\phi \in C^1(\mathbb{R}^d \times (0, T))$  be such that  $v^*(\bar{y}, \bar{s}) = \phi(\bar{y}, \bar{s})$  and  $v^* - \phi$  has a global strict maximum at  $(\bar{y}, \bar{s})$ . Since  $v^*(\cdot, \cdot)$  is upper semicontinuous, a standard argument in the theory of viscosity solutions implies that, up to some subsequence, there exists  $(y_n, s_n) \rightarrow (\bar{y}, \bar{s})$ , such that

$$(v_{\rho_n, h_n}[\mu_n] - \phi)(y_n, s_n) = \max_{(y, s) \in \mathbb{R}^d \times (0, T)} (v_{\rho_n, h_n}[\mu_n] - \phi)(y, s)$$

$$\text{and } (v_{\rho_n, h_n}[\mu_n] - \phi)(y_n, s_n) \rightarrow (v^* - \phi)(\bar{y}, \bar{s}) = 0.$$

Thus, for any  $(y, s) \in \mathbb{R}^d \times [0, T]$  we have that

$$v_{\rho_n, h_n}[\mu_n](y, s) \leq \phi(y, s) + \xi_n, \quad \text{with } \xi_n := (v_{\rho_n, h_n}[\mu_n] - \phi)(y_n, s_n) \rightarrow 0. \tag{4.14}$$



Let  $k := \mathbb{N} \rightarrow \{0, \dots, N-1\}$  be such that  $s_n \in [t_{k(n)}, t_{k(n)+1})$ . Evidently, we have that  $t_{k(n)} \rightarrow \bar{s}$ . By taking  $y = x_i$ ,  $i \in \mathbb{Z}^d$ , and  $s = t_{k(n)+1}$  in (4.14), we get that

$$v_{i,k(n)+1} \leq \phi(x_i, t_{k(n)+1}) + \xi_n \quad \text{for all } i \in \mathbb{Z}^d. \quad (4.15)$$

Lemma 4.1(ii)-(iii) implies that

$$S_{\rho_n, h_n}[\mu_n](v_{\cdot, k(n)+1}, i, k(n)) \leq S_{\rho_n, h_n}[\mu_n](\phi_{k(n)+1}, i, k(n)) + \xi_n \quad \text{for all } i \in \mathbb{Z}^d.$$

In particular, using (4.3), we get

$$v_{i,k(n)} \leq S_{\rho_n, h_n}[\mu_n](\phi_{k(n)+1}, i, k(n)) + \xi_n \quad \text{for all } i \in \mathbb{Z}^d,$$

which yields, by the definition of  $v_{\rho_n, h_n}[\mu_n](y_n, s_n)$  in (4.8),

$$v_{\rho_n, h_n}[\mu_n](y_n, s_n) \leq \sum_{i \in \mathbb{Z}^d} \beta_i(y_n) S_{\rho_n, h_n}[\mu_n](\phi_{k(n)+1}, i, k(n)) + \xi_n.$$

Now, recalling the definition of  $\xi_n$ , we get

$$\phi(y_n, s_n) \leq \sum_{i \in \mathbb{Z}^d} \beta_i(y_n) S_{\rho_n, h_n}[\mu_n](\phi_{k(n)+1}, i, k(n)). \quad (4.16)$$

We claim now that  $\phi(y_n, s_n) = \phi(y_n, t_{k(n)}) + O(h_n^2)$ . In fact, either  $s_n = t_{k(n)}$  (and the claim obviously holds), or  $s_n \in (t_{k(n)}, t_{k(n)+1})$ . In the latter case, since  $(v_{\rho_n, h_n} - \phi)(y_n, \cdot)$  has a maximum at  $s_n$  and  $v_{\rho_n, h_n}$  is constant in  $(t_{k(n)}, t_{k(n)+1})$ , then  $\partial_t \phi(y_n, s_n) = 0$  and the claim follows from a Taylor expansion. Thus, by our claim and (4.16), we have that

$$\phi(y_n, t_{k(n)}) \leq \sum_{i \in \mathbb{Z}^d} \beta_i(y_n) S_{\rho_n, h_n}[\mu_n](\phi_{k(n)+1}, i, k(n)) + o(h_n). \quad (4.17)$$

Now, inequality (4.17), estimate (4.2) and the fact that  $\rho_n^2/h_n \rightarrow 0$  imply that

$$\lim_{n \rightarrow \infty} \sum_{i \in \mathbb{Z}^d} \beta_i(y_n) \frac{\phi(x_i, t_{k(n)}) - S_{\rho_n, h_n}[\mu_n](\phi_{k(n)+1}, i, k(n))}{h_n} \leq 0.$$

Finally, by the consistency property in Lemma 4.1(iv) we obtain that

$$-\partial_t \phi(\bar{y}, \bar{s}) + \frac{1}{2} |D\phi(\bar{y}, \bar{s})|^2 - F(\bar{y}, \mu(\bar{s})) \leq 0,$$

which implies that  $v^*$  is a subsolution of (4.14). The supersolution property for  $v_*$  can be proved in a similar manner. Therefore, by a classical comparison argument,  $v_{\rho_n, h_n}[\mu_n]$  converges locally uniformly to  $v[\mu]$  in  $\mathbb{R}^d \times (0, T)$ . ■

Note that for all  $t \in [0, T]$ , the function  $v_{\rho, h}[\mu](\cdot, t)$  is not, in general, differentiable at  $x_i \in \mathcal{G}_\rho$ . Thus, we cannot use the useful characterizations of weak semiconcavity (see Lemma 4.2) for differentiable functions. Therefore, we will regularize  $v_{\rho, h}[\mu](\cdot, t)$  with the usual convolution technique. Let  $\rho \in C_c^\infty(\mathbb{R}^d)$  with  $\rho \geq 0$  and  $\int_{\mathbb{R}^d} \rho(x) dx = 1$ . For  $\varepsilon > 0$ , we consider the *mollifier*  $\rho_\varepsilon(x) := \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right)$  and define

$$v_{\rho, h}^\varepsilon[\mu](\cdot, t) := \rho_\varepsilon * v_{\rho, h}[\mu](\cdot, t) \quad \text{for all } t \in [0, T]. \quad (4.18)$$

Using Lemma 4.3(i) we easily check the estimates

$$\begin{aligned} \|v_{\rho, h}^\varepsilon[\mu](\cdot, \cdot) - v_{\rho, h}[\mu](\cdot, \cdot)\|_\infty &= \gamma\varepsilon, \\ \|Dv_{\rho, h}^\varepsilon[\mu](\cdot, \cdot)\|_\infty &= \frac{\gamma}{\varepsilon} \end{aligned} \quad (4.19)$$

where  $\gamma > 0$  is independent of  $(\varepsilon, \rho, h, \mu, t)$ . We have:

**Lemma 4.4** *For every  $t \in [0, T]$ , the following assertions hold true:*

- (i) [Lipschitz property] *The function  $v_{\rho, h}^\varepsilon[\mu](\cdot, t)$  is Lipschitz with constant  $d_0$  independent of  $(\rho, h, \mu, t)$ .*
- (ii) [Weak semiconcavity] *The function  $v_{\rho, h}^\varepsilon[\mu](\cdot, t)$  is  $(d_1, \rho)$ -weakly semiconcave, with  $d_1$  independent of  $(\rho, h, \varepsilon, \mu, t)$ .*

**Proof.** The result follows directly from the definition of  $v_{\rho, h}^\varepsilon[\mu](\cdot, t)$  and the corresponding results for  $v_{\rho, h}[\mu](\cdot, t)$  in Lemma 4.3. ■

As a consequence we obtain

**Theorem 4.2** *Let  $(\rho_n, h_n, \varepsilon_n) \rightarrow 0$  (as  $n \uparrow \infty$ ) be such that  $\frac{\rho_n^2}{h_n} \rightarrow 0$ . Then, for every sequence  $\mu_n \in C([0, T]; \mathcal{P}_1)$  such that  $\mu_n \rightarrow \mu$  in  $C([0, T]; \mathcal{P}_1)$ , we have that  $v_{\rho_n, h_n}^{\varepsilon_n}[\mu_n] \rightarrow v[\mu]$  uniformly over compact sets and  $Dv_{\rho_n, h_n}^{\varepsilon_n}[\mu_n](x, t) \rightarrow Dv[\mu](x, t)$  at every  $(x, t)$  such that  $Dv[\mu](x, t)$  exists.*

**Proof.** The first assertion is a consequence of Theorem 4.1 and the uniform estimate (4.19). Now, Lemma 4.4 and Lemma 4.2(iii) imply that  $Dv_{\rho_n, h_n}^{\varepsilon_n}[\mu_n](x, t)$  is uniformly bounded in  $n$  and

$$v_{\rho_n, h_n}^{\varepsilon_n}[\mu_n](y, t) - v_{\rho_n, h_n}^{\varepsilon_n}[\mu_n](x, t) - \langle Dv_{\rho_n, h_n}^{\varepsilon_n}[\mu_n](x, t), y - x \rangle \leq \frac{1}{2}d_1(|y - x|^2 + \rho_n^2).$$

Thus, every limit point  $p$  of  $Dv_{\rho_n, h_n}^{\varepsilon_n}[\mu_n](x, t)$  satisfies

$$v[\mu](y, t) - v[\mu](x, t) - \langle p, y - x \rangle \leq \frac{1}{2}d_1|y - x|^2 \quad \text{for all } x, y \in \mathbb{R}^d.$$

Therefore, if  $Dv[\mu](x, t)$  exists, the semiconcavity of  $v[\mu]$  implies that  $p = Dv[\mu](x, t)$  from which the result follows. ■

## 4.2 The fully-discrete scheme for the continuity equation

Given  $\mu \in C([0, T]; \mathcal{P}_1)$  and  $\varepsilon > 0$  let us define

$$\Phi_{i,k,k+1}^\varepsilon[\mu] := x_i - h\hat{\alpha}_{i,k}^\varepsilon[\mu] \quad \text{for all } i \in \mathbb{Z}^d, \quad k = 0, \dots, N-1, \quad (4.20)$$

where  $\hat{\alpha}_{i,k}^\varepsilon := \hat{\alpha}_{\rho,h}^\varepsilon[\mu](x_i, t_k)$  and  $\hat{\alpha}_{\rho,h}^\varepsilon[\mu] : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$  is defined as

$$\hat{\alpha}_{\rho,h}^\varepsilon[\mu](x, t) := Dv_{\rho,h}^\varepsilon[\mu](x, t). \quad (4.21)$$

Given the family  $\{\Phi_{i,k,k+1}^\varepsilon[\mu] ; i \in \mathbb{Z}^d, k = 0, \dots, N-1\}$ , we now consider a fully-discrete scheme for (2.15) which turns out to be equivalent to the one proposed [26], under some slight change of notation. Let us define

$$\mathcal{S} := \left\{ z = (z_i)_{i \in \mathbb{Z}^d} ; z_i \in \mathbb{R}_+ \text{ and } \sum_{i \in \mathbb{Z}^d} z_i = 1 \right\}.$$

The coordinates of  $m \in \mathcal{S}_{N+1} := \{\nu = (\nu_i)_{i=0}^N ; \nu_k \in \mathcal{S}\}$  are denoted as  $m_{i,k}$ , with  $i \in \mathbb{Z}^d$  and  $k = 0, \dots, N$ . We set

$$E_i := [x_i \pm \frac{1}{2}\rho e_1] \times \dots \times [x_i \pm \frac{1}{2}\rho e_d] \quad \text{for all } i \in \mathbb{Z}^d,$$

and define  $m^\varepsilon[\mu] \in \mathcal{S}_{N+1}$  recursively as

$$\begin{aligned} m_{i,k+1}^\varepsilon[\mu] &:= \sum_{j \in \mathbb{Z}^d} \beta_i(\Phi_{j,k,k+1}^\varepsilon[\mu]) m_{j,k}^\varepsilon[\mu], \quad \text{for } i \in \mathbb{Z}^d, \quad k = 0, \dots, N-1, \\ m_{i,0}^\varepsilon[\mu] &:= \int_{E_i} m_0(x) dx, \quad \text{for } i \in \mathbb{Z}^d. \end{aligned} \quad (4.22)$$

**Remark 4.1** Note that, omitting the dependence in  $\mu$ , for  $k = 0, \dots, N-1$  we have that

$$\sum_{i \in \mathbb{Z}^d} m_{i,k+1}^\varepsilon = \sum_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} \beta_i(\Phi_{j,k,k+1}^\varepsilon) m_{j,k}^\varepsilon = \sum_{j \in \mathbb{Z}^d} m_{j,k}^\varepsilon \sum_{i \in \mathbb{Z}^d} \beta_i(\Phi_{j,k,k+1}^\varepsilon) = \sum_{j \in \mathbb{Z}^d} m_{j,k}^\varepsilon = 1,$$

because  $\sum_{j \in \mathbb{Z}^d} m_{j,0}^\varepsilon = 1$ . Therefore, the scheme (4.22) is conservative.

Let us define  $m_{\rho,h}^\varepsilon[\mu] \in L^\infty(\mathbb{R}^d \times [0, T])$  as

$$\begin{aligned} m_{\rho,h}^\varepsilon[\mu](x, t) &:= \frac{1}{\rho^d} \left[ \frac{t_{k+1}-t}{h} \sum_{i \in \mathbb{Z}^d} m_{i,k}^\varepsilon[\mu] \mathbb{I}_{E_i}(x) + \frac{t-t_k}{h} \sum_{i \in \mathbb{Z}^d} m_{i,k+1}^\varepsilon[\mu] \mathbb{I}_{E_i}(x) \right], \\ &\text{if } t \in [t_k, t_{k+1}). \end{aligned} \quad (4.23)$$

Therefore, for every  $t \in [t_k, t_{k+1})$  we have

$$m_{\rho,h}^\varepsilon[\mu](x, t) := \left( \frac{t_{k+1}-t}{h} \right) m_{\rho,h}^\varepsilon[\mu](x, t_k) + \left( \frac{t-t_k}{h} \right) m_{\rho,h}^\varepsilon[\mu](x, t_{k+1}). \quad (4.24)$$

By abuse of notation, we continue to write  $m_{\rho,h}^\varepsilon[\mu](t)$  for the probability measure in  $\mathbb{R}^d$  whose density is given by (4.23). Thus, by the very definition, we can identify  $m_{\rho,h}^\varepsilon[\mu](\cdot, \cdot) \in L^\infty(\mathbb{R}^d \times [0, T])$  with an element  $m_{\rho,h}^\varepsilon[\mu](\cdot) \in C([0, T]; \mathcal{P}_1)$ .

We now study some technical properties of the family  $\{\Phi_{i,k,k+1}^\varepsilon[\mu] ; i \in \mathbb{Z}^d, k = 0, \dots, N-1\}$ . The next result, which is an easy consequence of the weak semiconcavity of  $v_{\rho,h}^\varepsilon[\mu]$ , is similar to the one proved in Proposition 3.2.

**Proposition 4.1** *For any  $i, j \in \mathbb{Z}^d$  and  $k = 0, \dots, N-1$ , we have*

$$|\Phi_{i,k,k+1}^\varepsilon[\mu] - \Phi_{j,k,k+1}^\varepsilon[\mu]|^2 \geq (1 - d_2 h) |x_i - x_j|^2 - d_2 h \rho^2, \quad (4.25)$$

where  $d_2 \geq 0$  is independent of  $(\rho, h, \varepsilon, \mu)$ .

**Proof.** For the reader's convenience, we omit the  $\mu$  argument. Recalling (4.20) and (4.21), for every  $k = 0, \dots, N-1$  we have

$$\begin{aligned} |\Phi_{i,k,k+1}^\varepsilon - \Phi_{j,k,k+1}^\varepsilon|^2 &= \left| x_i - x_j - h \left[ Dv_{\rho,h}^\varepsilon(x_i, t_k) - Dv_{\rho,h}^\varepsilon(x_j, t_k) \right] \right|^2, \\ &= |x_i - x_j|^2 + h^2 |Dv_{\rho,h}^\varepsilon(x_i, t_k) - Dv_{\rho,h}^\varepsilon(x_j, t_k)|^2 + \\ &\quad - 2h \langle Dv_{\rho,h}^\varepsilon(x_i, t_k) - Dv_{\rho,h}^\varepsilon(x_j, t_k), x_i - x_j \rangle, \end{aligned}$$

which yields to

$$|\Phi_{i,k,k+1}^\varepsilon - \Phi_{j,k,k+1}^\varepsilon|^2 \geq |x_i - x_j|^2 - 2h \langle Dv_{\rho,h}^\varepsilon(x_i, t_k) - Dv_{\rho,h}^\varepsilon(x_j, t_k), x_i - x_j \rangle.$$

Therefore, by Lemma 4.4(ii) and (4.12) in Lemma 4.2, there exists  $d_2 > 0$  such that (4.25) holds. ■

Now we provide a technical result which, in the case  $d = 1$ , allow us to obtain uniform  $L^\infty$  bounds for  $m_{\rho,h}^\varepsilon[\mu]$  (see Proposition 4.2(ii) below).

**Lemma 4.5** *Suppose that  $d = 1$ . Then, there exists  $d_3 > 0$  (independent of  $h$  small enough and  $(\rho, \varepsilon, \mu)$ ) such that for any  $i \in \mathbb{Z}$  and  $k = 0, \dots, N-1$ , we have that*

$$\sum_{j \in \mathbb{Z}} \beta_i(\Phi_{j,k,k+1}^\varepsilon) \leq 1 + d_3 h. \quad (4.26)$$

**Proof.** For notational simplicity, let us set  $y_j = \Phi_{j,k,k+1}^\varepsilon$ . Note that for any  $j_1, j_2 \in \mathbb{Z}$ , Proposition 4.1 implies that

$$|y_{j_1} - y_{j_2}|^2 \geq (1 - d_2 h) |x_{j_1} - x_{j_2}|^2 - d_2 h \rho^2.$$

Thus, if  $j_1 \neq j_2$ , we get

$$|y_{j_1} - y_{j_2}|^2 \geq (1 - 2d_2 h) \rho^2, \quad \text{i.e. } |y_{j_1} - y_{j_2}| \geq \sqrt{(1 - 2d_2 h)} \rho. \quad (4.27)$$

Since the diameter of  $\text{supp}(\beta_i)$  is equal to  $2\rho$ , the above inequality implies that for  $h$  small enough (independent of  $(\rho, \varepsilon, \mu)$ ), the cardinality of

$$\mathcal{Z}_i := \{j \in \mathbb{Z} ; y_j \in \text{supp}(\beta_i)\}$$

is at most 3. If  $\mathcal{Z}_i$  only has one element, then (4.26) is trivial. If  $\mathcal{Z}_i$  has two elements  $y_{j_1}, y_{j_2}$  with  $y_{j_1} < y_{j_2}$ , then

$$\beta_i(y_{j_1}) + \beta_i(y_{j_2}) = 2 - \frac{|y_{j_1} - x_i|}{\rho} - \frac{|y_{j_2} - x_i|}{\rho} \leq 2 - \frac{|y_{j_1} - y_{j_2}|}{\rho},$$

by the triangular inequality. Using (4.27) we get

$$\beta_i(y_{j_1}) + \beta_i(y_{j_2}) \leq 2 - \sqrt{(1 - 2d_2h)} \leq 1 + 2d_2h,$$

from which (4.27) follows. Finally, if  $\mathcal{Z}_i$  has three elements  $y_{j_1}, y_{j_2}$  and  $y_{j_3}$ , then (supposing for example that  $y_{j_1} \leq y_{j_2} \leq x_i < y_{j_3}$ ) we have

$$\begin{aligned} \beta_i(y_{j_1}) + \beta_i(y_{j_3}) &= 1 - \frac{x_i - y_{j_1}}{\rho} + 1 - \frac{y_{j_3} - x_i}{\rho}, \\ &= 2 - \frac{y_{j_2} - y_{j_1}}{\rho} - \frac{y_{j_3} - y_{j_2}}{\rho} \leq 2 - 2\sqrt{(1 - 2d_2h)} \leq 4d_2h. \end{aligned}$$

Using that  $\beta_i(y_{j_2}) \leq 1$  and the above estimate, we obtain (4.27) with  $d_3 := 4d_2$ . ■

Using the above results, we can establish some important properties for  $m_{\rho,h}^\varepsilon[\mu]$ , which are similar to those found for  $m_h[\mu]$  in the semi-discrete case (see Proposition 3.3).

**Proposition 4.2** *Suppose that  $\rho = O(h)$ . Then, there exists a constant  $d_4 > 0$  (independent of  $(\rho, h, \varepsilon, \mu)$ ) such that:*

(i) *For all  $t_1, t_2 \in [0, T]$ , we have that*

$$\mathbf{d}_1(m_{\rho,h}^\varepsilon[\mu](t_1), m_{\rho,h}^\varepsilon[\mu](t_2)) \leq d_4|t_1 - t_2|. \quad (4.28)$$

(ii) *For all  $t \in [0, T]$ ,  $m_{\rho,h}^\varepsilon[\mu](t)$  has a support in  $B(0, d_4)$ .*

(iii) *If  $d = 1$  then we have*

$$\|m_{\rho,h}^\varepsilon[\mu](\cdot, t)\|_\infty \leq d_4.$$

**Proof.** Let  $\phi \in C(\mathbb{R}^d)$  be a 1-Lipschitz function. By (4.24), the function  $\psi_\phi : [0, T] \rightarrow \mathbb{R}$ , defined as

$$\psi_\phi(t) := \int_{\mathbb{R}^d} \phi(x) dm_{\rho,h}^\varepsilon[\mu](t),$$

is affine in each interval  $[t_k, t_{k+1}]$ , with  $k = 0, \dots, N-1$ . It clearly belongs to  $W^{1,\infty}([0, T])$  and

$$\left\| \frac{d}{dt} \psi_\phi \right\|_\infty = \frac{1}{h} \max_{k=0, \dots, N-1} \left| \int_{\mathbb{R}^d} \phi(x) d[m_{\rho,h}^\varepsilon[\mu](t_{k+1}) - m_{\rho,h}^\varepsilon[\mu](t_k)] \right|.$$

For every  $k = 0, \dots, N-1$  we have, omitting  $\mu$  from the notation,

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x) d[m_{\rho,h}^\varepsilon(t_{k+1}) - m_{\rho,h}^\varepsilon(t_k)] &= \frac{1}{\rho^d} \sum_{i \in \mathbb{Z}^d} \int_{E_i} \phi(x) dx \left[ \sum_{j \in \mathbb{Z}^d} \beta_i(\Phi_{j,k,k+1}^\varepsilon) m_{j,k}^\varepsilon - m_{i,k}^\varepsilon \right], \\ &= \sum_{j \in \mathbb{Z}^d} m_{j,k}^\varepsilon \left[ \sum_{i \in \mathbb{Z}^d} \beta_i(\Phi_{j,k,k+1}^\varepsilon) \frac{1}{\rho^d} \int_{E_i} \phi(x) dx - \frac{1}{\rho^d} \int_{E_j} \phi(x) dx \right]. \end{aligned}$$

On the other hand, since  $\phi$  is 1-Lipschitz, we have that

$$\left| \frac{1}{\rho^d} \int_{E_i} \phi(x) dx - \phi(x_i) \right| \leq \rho. \quad (4.29)$$

Using (4.29), estimate (4.2), Lemma 4.4(i) and the fact that  $\rho = O(h)$ , we get that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \phi(x) d[m_{\rho,h}^\varepsilon(t_{k+1}) - m_{\rho,h}^\varepsilon(t_k)] \right| &\leq \sum_{j \in \mathbb{Z}^d} m_{j,k}^\varepsilon \left| \sum_{i \in \mathbb{Z}^d} \beta_i(\Phi_{j,k,k+1}^\varepsilon) \phi(x_i) - \phi(x_j) \right| + 2\rho, \\ &= \sum_{j \in \mathbb{Z}^d} m_{j,k}^\varepsilon \left| \phi(\Phi_{j,k,k+1}^\varepsilon) - \phi(x_j) \right| + 2c\rho, \\ &\leq d_0 h + 2c\rho = (d_0 + \frac{2c\rho}{h}) h \leq c' h, \end{aligned}$$

for some constants  $c, c' > 0$  independent of  $(\rho, h, \varepsilon, \mu)$ . Therefore, we obtain that  $\left\| \frac{d}{dt} \psi_\phi \right\|_\infty \leq c'$ , which proves (i) with  $d_4$  to be chosen later.

In order to prove (ii), it suffices to note that since  $\|Dv_{\rho,h}^\varepsilon[\mu]\|_\infty \leq d_0$  we easily check that  $\text{supp}(m_{\rho,h}^\varepsilon[\mu](t)) \subset B(0, c_1 + 2d_0 T)$ . Now, let us assume  $d = 1$ . By the definition of  $m_{\rho,h}^\varepsilon[\mu](\cdot, 0)$  in (4.23) and assumption **(H1)**, we have

$$\|m_{\rho,h}^\varepsilon[\mu](\cdot, 0)\|_\infty = \max_{i \in \mathbb{Z}} \left\{ \frac{1}{\rho} m_{i,0}^\varepsilon[\mu] \right\} \leq \|m_0\|_\infty \leq c_1.$$

Now, given  $k = 0, \dots, N-1$ , we have that

$$\|m_{\rho,h}^\varepsilon[\mu](\cdot, t_{k+1})\|_\infty \leq \max_{i \in \mathbb{Z}} \left\{ \frac{1}{\rho} m_{i,k+1}^\varepsilon[\mu] \right\} = \frac{1}{\rho} \max_{i \in \mathbb{Z}} \left\{ \sum_{j \in \mathbb{Z}} \beta_i(\Phi_{j,k,k+1}^\varepsilon) m_{j,k}^\varepsilon[\mu] \right\}.$$

Therefore, by Lemma 4.5, we obtain that

$$\|m_{\rho,h}^\varepsilon[\mu](\cdot, t_{k+1})\|_\infty \leq \|m_{\rho,h}^\varepsilon[\mu](\cdot, t_k)\|_\infty \sum_{j \in \mathbb{Z}} \beta_j(\Phi_{j,k,k+1}^\varepsilon) \leq (1 + d_3 h) \|m_{\rho,h}^\varepsilon[\mu](\cdot, t_k)\|_\infty.$$

Iterating in the above expression, we obtain that

$$\|m_{\rho,h}^\varepsilon[\mu](\cdot, t_{k+1})\|_\infty \leq (1 + d_3 h)^{\frac{T}{h}} \|m_0\|_\infty \leq e^{d_3 T} c_1,$$

for  $h$  small enough. The result follows, by taking  $d_4 = \max\{c', c_1 + 2d_0 T, e^{d_3 T} c_1\}$ . ■

### 4.3 The fully-discrete scheme for the first order MFG problem (1.2)

For a given  $\rho, h, \varepsilon > 0$  and  $\mu \in \mathcal{S}^{N+1}$  we still write  $\mu$  for the element in  $C([0, T]; \mathcal{P}_1)$  defined as

$$\mu(x, t) := \frac{1}{\rho^d} \left[ \frac{t_{k+1} - t}{h} \sum_{i \in \mathbb{Z}^d} \mu_{i,k} \mathbb{I}_{E_i}(x) + \frac{t - t_k}{h} \sum_{i \in \mathbb{Z}^d} \mu_{i,k+1} \mathbb{I}_{E_i}(x) \right] \text{ if } t \in [t_k, t_{k+1}]. \quad (4.30)$$

Let us consider the following *fully-discretization* of (MFG):

$$\text{Find } \mu \in \mathcal{S}_{N+1} \text{ such that } \mu_{i,k} = m_{i,k}^\varepsilon[\mu] \quad \forall i \in \mathbb{Z}^d \text{ and } k = 0, \dots, N, \quad (4.31)$$

where we recall that  $m_{i,k}^\varepsilon[\mu]$  is defined in (4.22). In order to prove that (4.31) admits at least a solution, we will need the following stability result.

**Lemma 4.6** *Let  $\mu^n \in \mathcal{S}_{N+1}$  be a sequence converging to  $\mu \in \mathcal{S}_{N+1}$ . Then:*

- (i)  $v_{\rho,h}^\varepsilon[\mu^n](\cdot, \cdot) \rightarrow v_{\rho,h}^\varepsilon[\mu](\cdot, \cdot)$  uniformly over compact sets.
- (ii)  $m_{i,k}^\varepsilon[\mu^n] \rightarrow m_{i,k}^\varepsilon[\mu]$  for all  $i \in \mathbb{Z}^d$  and  $k = 0, \dots, N$ .

**Proof.** Because of the assumptions on  $F$  and  $G$  in **(H1)** we clearly have (i). By definition of  $v_{\rho,h}^\varepsilon[\mu^n](x, t)$  and (i), Lebesgue theorem implies that we have pointwise convergence of  $Dv_{\rho,h}^\varepsilon[\mu^n]$  to  $Dv_{\rho,h}^\varepsilon[\mu]$  and obviously also of  $\hat{\alpha}_\varepsilon^{\rho,h}[\mu^n](\cdot, \cdot) \rightarrow \hat{\alpha}_\varepsilon^{\rho,h}[\mu](\cdot, \cdot)$ . Assertion (ii) for  $i \in \mathbb{Z}^d$  and  $k = 1$  follows hence from the definition (4.22) of  $m_{i,1}^\varepsilon[\mu^n]$ . Therefore, by recursive argument we get the result for all  $i \in \mathbb{Z}^d$  and  $k = 0, \dots, N-1$ . ■

**Theorem 4.3** *There exists at least one solution of (4.31).*

**Proof.** This is a straightforward consequence of Lemma 4.6, Proposition 4.2(ii) and Brouwer fixed-point theorem. ■

Given a solution  $m^\varepsilon \in \mathcal{S}_{N+1}$  of (4.31), we set  $m_{\rho,h}^\varepsilon(\cdot, \cdot)$  for the extension to  $\mathbb{R}^d \times [0, T]$  defined in (4.23).

Now we prove our main result.

**Theorem 4.4** *Suppose that  $d = 1$  and that **(H1)**-**(H3)** hold. Consider a sequence of positive numbers  $\rho_n, h_n, \varepsilon_n$  satisfying that  $\rho_n = o(h_n)$ ,  $h_n = o(\varepsilon_n)$  and  $\varepsilon_n \downarrow 0$  as  $n \uparrow \infty$ . Let  $\{m^n\}_{n \in \mathbb{N}}$  be a sequence of solutions of (4.31) for the corresponding parameters  $\rho_n, h_n, \varepsilon_n$ . Then every limit point in  $C([0, T]; \mathcal{P}_1)$  of  $m^n$  (there exists at least one) solves (MFG). In particular, if **(H4)** holds we have that  $m_{\rho_n, h_n}^{\varepsilon_n} \rightarrow m$  (the unique solution of (MFG)) in  $C([0, T]; \mathcal{P}_1)$  and in  $L^\infty(\mathbb{R}^d \times [0, T])$ -weak- $*$ .*

**Proof.** For notational convenience we will write  $v^n := v_{\rho_n, h_n}^{\varepsilon_n}[m^n]$ . By Proposition 4.2(i) and Ascoli theorem we can assume the existence of  $\bar{m} \in C([0, T]; \mathcal{P}_1)$  such that  $m^n$  (as an element of  $C([0, T]; \mathcal{P}_1)$ ) converge to  $\bar{m}$  in  $C([0, T]; \mathcal{P}_1)$ . Moreover, Proposition 4.2(iii) implies that, up to some subsequence,  $m^n$  (as an element of  $L^\infty(\mathbb{R}^d \times [0, T])$ ) converge in  $L^\infty(\mathbb{R}^d \times [0, T])$ -weak-\* to some  $\hat{m}$ . Thus, we necessarily have that  $\bar{m}$  is absolutely continuous and its density, still denoted as  $\bar{m}$ , is equal to  $\hat{m}$ . In order to complete the proof, we now show that  $\bar{m}$  solves the continuity equation (2.3), i.e. for any  $t \in [0, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}} \phi(x) d\bar{m}(t)(x) = \int_{\mathbb{R}} \phi(x) dm_0(x) - \int_0^t \int_{\mathbb{R}} D\phi(x) Dv[\bar{m}](x, s) d\bar{m}(s)(x) ds. \quad (4.32)$$

Given  $t \in [0, T]$ , let us set  $t_n := \left\lfloor \frac{t}{h_n} \right\rfloor h_n$ . We have

$$\int_{\mathbb{R}} \phi(x) dm^n(t_n) = \int_{\mathbb{R}} \phi(x) dm_0(x) + \sum_{k=0}^{n-1} \int_{\mathbb{R}} \phi(x) d[m^n(t_{k+1}) - m^n(t_k)]. \quad (4.33)$$

By definitions (4.22) and (4.23), setting  $\Phi_{i,k,k+1}^n := x_i - h_n Dv^n(x_i, t_k)$ , for all  $k = 0, \dots, n-1$  we have

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) dm^n(t_{k+1}) &= \sum_{i \in \mathbb{Z}} m_{i,k+1}^n \frac{1}{\rho_n} \int_{E_i} \phi(x) dx, \\ &= \sum_{i \in \mathbb{Z}} \frac{1}{\rho_n} \int_{E_i} \phi(x) dx \sum_{j \in \mathbb{Z}} \beta_i \left( \Phi_{j,k,k+1}^n \right) m_{j,k}^n, \\ &= \sum_{j \in \mathbb{Z}} m_{j,k}^n \sum_{i \in \mathbb{Z}} \beta_i \left( \Phi_{j,k,k+1}^n \right) \frac{1}{\rho_n} \int_{E_i} \phi(x) dx. \end{aligned} \quad (4.34)$$

As in (4.29) we get

$$\left| \frac{1}{\rho} \int_{E_i} \phi(x) dx - \phi(x_i) \right| \leq \|D\phi\|_\infty \rho.$$

Therefore, combining with (4.34), we get (recalling (4.2) with  $\gamma = 1$ )

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) dm^n(t_{k+1}) &= \sum_{j \in \mathbb{Z}} m_{j,k}^n \sum_{i \in \mathbb{Z}} \beta_i \left( \Phi_{j,k,k+1}^n \right) \phi(x_i) + O(\rho), \\ &= \sum_{j \in \mathbb{Z}} m_{j,k}^n I[\phi] \left( \Phi_{j,k,k+1}^n \right) + O(\rho), \\ &= \sum_{j \in \mathbb{Z}} m_{j,k}^n \phi \left( \Phi_{j,k,k+1}^n \right) + O(\rho). \end{aligned} \quad (4.35)$$

On the other hand, by Lemma 4.3(i), the function  $v^n(\cdot, t)$  is Lipschitz (with Lipschitz constant independent of  $n$ ). Therefore, by (4.19) we have the existence of a constant  $c > 0$  (independent of  $n$ ) such that

$$|Dv^n(x, t) - Dv^n(y, t)| \leq \frac{c}{\varepsilon_n} |x - y|, \quad (4.36)$$



which implies, setting  $\Phi_{k,k+1}^n(x) = x - h_n Dv^n(x, t)$ , that

$$|\phi(\Phi_{k,k+1}^n(x)) - \phi(\Phi_{k,k+1}^n(y))| \leq c' \left(1 + \frac{h}{\varepsilon_n}\right) |x - y|.$$

for some  $c' > c$  (which is also independent of  $n$ ). Therefore, we have

$$\left| \frac{1}{\rho} \int_{E_j} \phi(\Phi_{k,k+1}^n(x)) dx - \phi(\Phi_{j,k,k+1}^n) \right| \leq c' \left(1 + \frac{h}{\varepsilon_n}\right) \rho.$$

Since  $\frac{h}{\varepsilon_n} = O(1)$ , by (4.35), we get

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) dm^n(t_{k+1}) &= \sum_{j \in \mathbb{Z}} m_{j,k}^n \frac{1}{\rho_n} \int_{E_j} \phi(\Phi_{k,k+1}^n(x)) dx + O(\rho), \\ &= \int_{\mathbb{R}} \phi(\Phi_{k,k+1}^n(x)) dm^n(t_k) + O(\rho). \end{aligned}$$

The expression above yields to

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) d[m^n(t_{k+1}) - m^n(t_k)] &= \int_{\mathbb{R}} [\phi(\Phi_{k,k+1}^n(x)) - \phi(x)] dm^n(t_k) + O(\rho), \\ &= -h_n \int_{\mathbb{R}} D\phi(x) Dv^n(x, t_k) dm^n(t_k) \\ &\quad + O(h_n^2 + \rho_n) \end{aligned} \tag{4.37}$$

Since  $D\phi(\cdot) \cdot Dv^n(\cdot, t_k)$  is  $c''/\varepsilon_n$ -Lipschitz (with  $c''$  large enough), Proposition 4.2(i) gives that for all  $s \in [t_k, t_{k+1}]$ , with  $k = 0, \dots, n-1$ , we have

$$\left| \int_{\mathbb{R}} D\phi(x) Dv^n(x, t_k) d[m^n(s) - m^n(t_k)] \right| \leq \frac{c''}{\varepsilon_n} |s - t_k| \leq \frac{c'' h_n}{\varepsilon_n},$$

which implies that, using that  $Dv^n(x, s) = Dv^n(x, t_k)$  for  $s \in [t_k, t_{k+1}]$ ,

$$\left| \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}} D\phi(x) Dv^n(x, s) d[m^n(s) - m^n(t_k)] ds \right| \leq \frac{c'' h_n^2}{\varepsilon_n}. \tag{4.38}$$

Therefore, combining (4.38) and (4.37), we obtain that

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) d[m^n(t_{k+1}) - m^n(t_k)] &= - \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}} D\phi(x) Dv^n(x, s) dm^n(s)(x) ds \\ &\quad + O\left(\frac{h_n^2}{\varepsilon_n} + \rho_n\right). \end{aligned}$$

Thus, summing from  $k = 0$  to  $k = n-1$  and using (4.33)

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) dm^n(t_n)(x) &= \int_{\mathbb{R}} \phi(x) m^n(x, 0) - \int_0^{t_n} \int_{\mathbb{R}} D\phi(x) Dv^n(x, s) m^n(x, s) dx ds \\ &\quad + O\left(\frac{h_n}{\varepsilon_n} + \frac{\rho_n}{h_n}\right). \end{aligned} \tag{4.39}$$

By Theorem 4.2 we have that  $Dv^n(x, s) \rightarrow Dv[\bar{m}](x, s)$  for a.a.  $(x, s) \in \mathbb{R} \times [0, T]$ . Therefore, using that  $\phi \in C_c^\infty(\mathbb{R})$ , the Lebesgue theorem implies that

$$\mathbb{I}_{[0, t_n]} D\phi(\cdot) \cdot Dv^n(\cdot, \cdot) \rightarrow \mathbb{I}_{[0, t]} D\phi(\cdot) \cdot Dv[\bar{m}](\cdot, \cdot) \in L^1(\mathbb{R} \times [0, T]) \text{ strongly in } L^1,$$

and since  $m^n$  converge to  $\bar{m}$  in  $L^\infty(\mathbb{R} \times [0, T])$ -weak-\*, we can pass to the limit in (4.39) to obtain (4.32). The result follows. ■

## 5 Numerical Tests

We show numerical simulations for the case  $d = 1$ . Given  $\varepsilon, \rho, h > 0$  we set  $\{m_{i,k}^\varepsilon; i \in \mathbb{Z}^d, k = 0, \dots, [\frac{T}{h}]\}$  for the solution of (4.31) and  $\{v_{i,k}^\varepsilon; i \in \mathbb{Z}^d, k = 0, \dots, [\frac{T}{h}]\}$  for the associate value functions. We approximate heuristically  $m_{i,k}^\varepsilon$  and  $v_{i,k}^\varepsilon$  with a fixed-point iteration method. We consider as initial guess the element in  $m^{\varepsilon,0} \in \mathcal{S}_{N+1}$  given by

$$m_{i,k}^{\varepsilon,0} = m_{i,0}^\varepsilon = \int_{E_i} m_0(x) dx, \quad i \in \mathbb{Z}, k = 0, \dots, N.$$

Next, for  $p = 0, 1, 2, \dots$ , given  $m^{\varepsilon,p} \in \mathcal{S}_{N+1}$  we calculate  $v^{\varepsilon,p+1} \in B(\mathcal{G}_{\rho,h})$  with the backward scheme (4.3), taking as  $\mu$  the extension of  $m^{\varepsilon,p}$  to  $C([0, T]; \mathcal{P}_1)$  defined in (4.30). The element  $m^{\varepsilon,p+1} \in \mathcal{S}_{N+1}$  is then computed with the forward scheme (4.22), taking

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \quad (5.1)$$

In the numerical simulations we approximate (4.21) with a discrete convolution using a central difference scheme for the gradient. The iteration process is stopped once the quantities

$$E(v^{\varepsilon,p}) := \|v^{\varepsilon,p+1} - v^{\varepsilon,p}\|_\infty, \quad E(m^{\varepsilon,p}) := \|m^{\varepsilon,p+1} - m^{\varepsilon,p}\|_\infty, \quad (5.2)$$

are below a given threshold  $\tau$ .

**Remark 5.1** *The theoretical study of the convergence of the fixed-point iterations is not analyzed in the present paper. The analysis of a convergent and efficient method to solve (4.31) remains as subject of future research.*

By Proposition 4.2(ii), we know that  $m^\varepsilon$  has a compact support, uniformly in  $(\varepsilon, \rho, h)$ . Therefore, in order to calculate the iteration  $m_{i,k}^{\varepsilon,p+1}$  we only need the values  $v_{i,k}^{\varepsilon,p+1}$  for  $i$  such that  $i\rho$  belongs to a compact set  $K$ , which is independent of  $(\varepsilon, \rho, h, p)$ . This fact allows us to drop the analysis of boundary conditions.

For the numerical tests we will consider running costs of the form

$$\frac{1}{2}\alpha^2(t) + F(x, m(t)) = \frac{1}{2}\alpha^2(t) + f(x) + V(x, m(t)),$$

where  $f$  is  $C^2$  and

$$V(x, m(t)) = \rho_\sigma * [\rho_\sigma * m(t)](x), \quad \text{for some } \sigma > 0 \text{ to be chosen later.} \quad (5.3)$$

A straightforward calculation shows that  $F(x, m(t)) = f(x) + V(x, m(t))$  satisfies assumption **(H4)**.

### 5.1 Test 1

We simulate a game where the agents are adverse to the presence of other agents during the game and, at the end, they do not want to live at the boundary of a domain  $\Omega$ .

In order to model this situation, we take  $\Omega = [-0.2, 1.2]$ , and the running cost

$$\frac{1}{2}\alpha^2 + F(x, m) = \frac{1}{2}\alpha^2 + 0.3V(x, m),$$

where  $V$  given by (5.3) with  $\sigma = 0.3$ . We choose  $T = 1$  as final time and

$$G(x) = -0.5(x + 0.5)^2(1.5 - x)^2,$$

as final cost function. We take as initial mass distribution

$$m_0(x) = \frac{\nu(x)}{\int_{\Omega} \nu(x) dx}$$

where  $\nu(x) = \mathbb{I}_{[0,1]}(x)(1 - 0.2\cos(\pi x))$ . We choose  $\varepsilon = 0.1$ , as space step  $\rho = 1.75 \cdot 10^{-2}$  and as time step  $h = 0.02$ .

The function  $F$  penalizes high mass density during the game whereas the final condition  $G$  penalizes the fact that the agents are near the boundary at time  $T$ . In Fig.1, we show the mass evolution in the time-space domain  $\Omega \times [0, T]$ . It is possible to observe that from the initial configuration, the mass distribution moves where the final cost is lower and at the same time it does not accumulate completely at the center.

In Fig. 2 the discrete value function  $v_{i,k}^\varepsilon$  and in Fig. 3 the gradient  $Dv_{i,k}^\varepsilon$  are shown in the domain  $\Omega \times [0, T]$ . Fig. 4 shows the behavior of the errors (5.2) in logarithmic scale on the  $y$ -axis with respect to number of fixed-point iterations on the  $x$ -axis. The fixed point iteration method has been stopped when both errors are below  $\tau = 10^{-3}$ .

### 5.2 Test 2

We model now a game where the agents want to live at  $x = 0.2$  but again they are adverse to the presence of other agents. We consider a space numerical domain given by  $\Omega = [0, 1]$  and a final time  $T = 1$ . The running cost function is modeled as

$$\frac{1}{2}\alpha^2 + F(x, m) = \frac{1}{2}\alpha^2 + (x - 0.2)^2 + V(x, m),$$

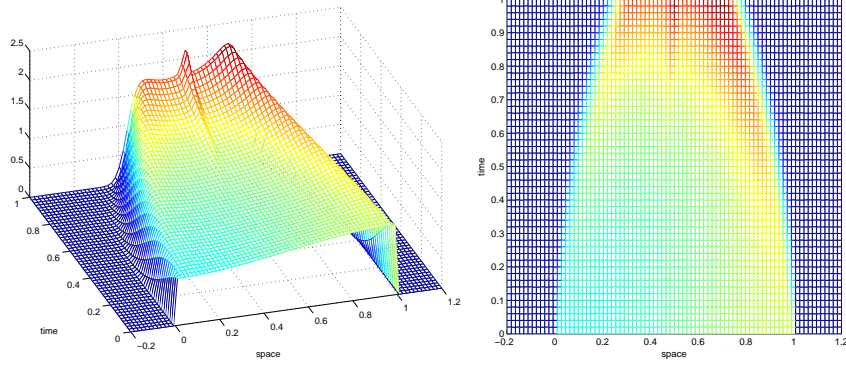


Figure 1: **Mass evolution**  $m_{i,k}^\varepsilon$

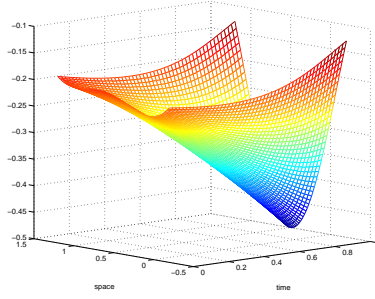


Figure 2: **Value function**  $v_{i,k}^\varepsilon$

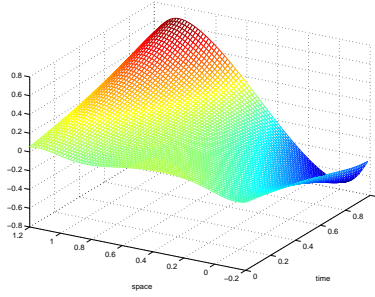


Figure 3: **Gradient**  $Dv_{i,k}^\varepsilon$

where  $V(x, m)$  is defined in (5.3) with  $\sigma = 0.05$ . We do not consider a final cost, i.e. we take  $G \equiv 0$ . We choose as initial mass distribution:

$$m_0(x) = \frac{\nu(x)}{\int_{\Omega} \nu(x) dx}, \text{ with } \nu(x) = e^{-(x-0.75)^2/(0.1)^2}.$$

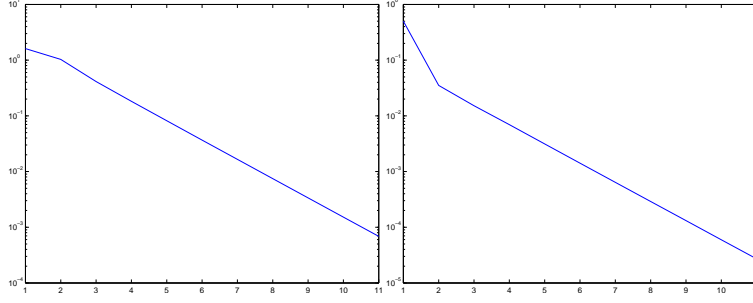


Figure 4: **Errors:**  $E(m^{\varepsilon,p})$  (left),  $E(v^{\varepsilon,p})$  (right),  $p = 0, \dots, 10$

We choose  $\varepsilon = 0.2$ , as space discretization step  $\rho = 1.25 \cdot 10^{-2}$  and as time step  $h = 0.02$ .

Fig. 5 shows the mass evolution. As it is expected, during the evolution the mass

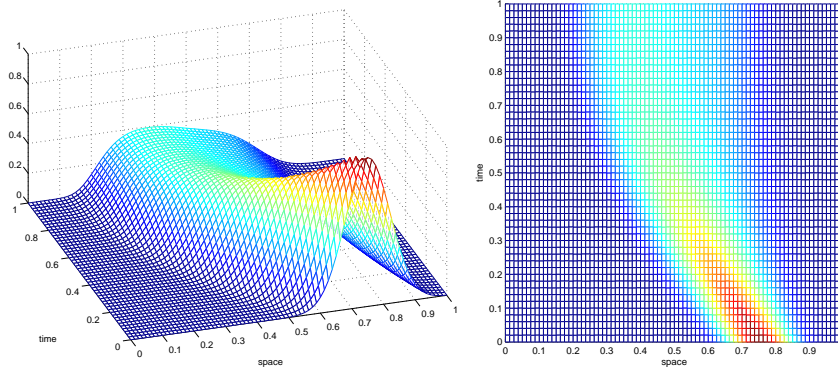


Figure 5: **Mass distribution**  $m_{i,k}^{\varepsilon}$

distribution tends to concentrate at the “low energy” configuration  $x = 0.2$  but at the same time the second term in  $F$  penalize this concentration. In Fig.6 the discrete value function is shown and in Fig. 7 we display its gradient.

In Fig. 8, we show the errors  $E(m^{\varepsilon,p})$  and  $E(v^{\varepsilon,p})$  of the fixed-point algorithm for the mass and the value function. The fixed-point iteration has been stopped when both the errors are below  $\tau = 10^{-3}$ .

Let us finally compare this test to the case when there is no game, i.e. the running cost does not depend on  $m$ :

$$F(x, m) = (x - 0.2)^2.$$

In this case, the system is not coupled and after one iteration we obtain the solution.

In Fig. 9, the mass evolution is shown. It is seen that, during the evolution, the measure

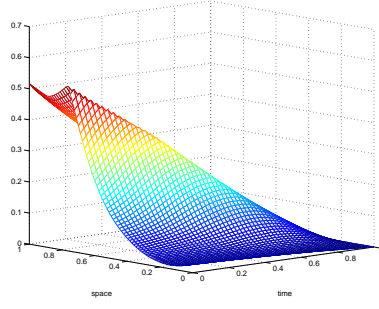


Figure 6: **Value function**  $v_{i,k}^\varepsilon$

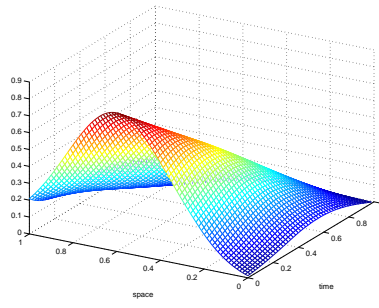


Figure 7: **Gradient**  $Dv_{i,k}^\varepsilon$

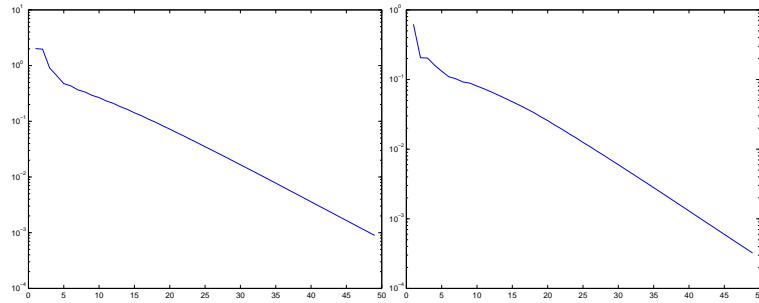


Figure 8: **Errors:**  $E(m^{\varepsilon,p})$  (left),  $E(v^{\varepsilon,p})$  (right),  $p = 0, \dots, 50$

maintains its original shape, due to the absence of conflict between the agents. This shows qualitative differences with the situation where conflict is present, as was displayed in the case of Fig. 5.

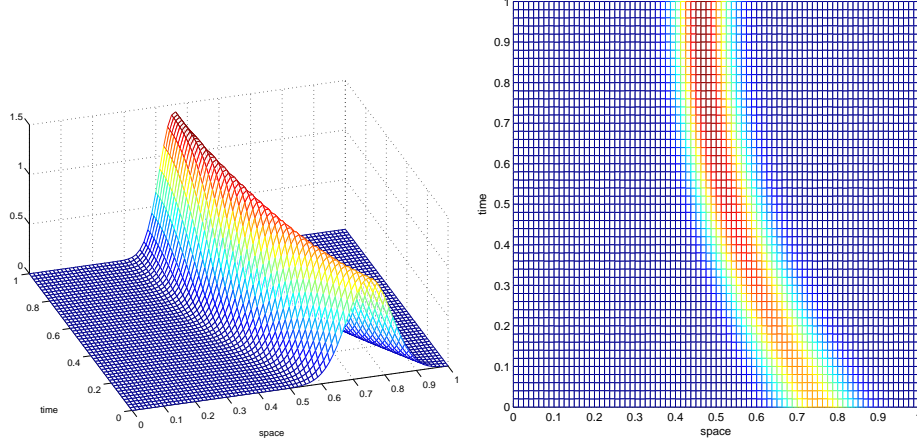


Figure 9: Mass distribution  $m_{i,k}^\varepsilon$  (case “no game” with  $F = (x - 0.2)^2$ )

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